Methods of Estimation of Structure Borne Noise in Structures - Review


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METHODS OF ESTIMATION OF STRUCTURE BORNE NOISE IN STRUCTURES - REVIEW

MARCM WACHULEC, POUL H. KIRKEGAARD, SØREN R.K. NIELSEN

Abstract. This paper gives an overview of the methods used to estimate the structure borne noise. Of main interest are the methods applied to medium frequency range. First the methods used in low and high frequency ranges are presented and their limitations are described. Then the works aiming to overcome those limitations are presented. This review is a part of a research programme focused on structure borne noise generated by steel railway bridges.

1. Introduction
The prediction of sound radiation from vibrating structures becomes important in the process of design and development. The vibroacoustic analysis consists of dynamic analysis of the structure, and the sound radiation from it. It is often assumed in structural engineering that the 'back loop' - the influence of the fluid on structure is negligible and may be thus omitted. This assumption allows a two-stage analysis, where the vibration of structure is treated first, and the sound radiation - second. Thus the accuracy of the analysis strongly depends on the accuracy of structural vibration[4, 12] analysis. There is, however, some uncertainty if the analysis is performed in the previously mentioned manner, thus there are attempts to analyse both vibration and radiation. Those methods will be presented in the latter part.

If one attempt to analyse the dynamic behaviour of the structure a mathematical model is required. It is done by setting up the differential equation of motion. Then one has to consider the propagation of disturbances through the structure. This is known as wave approach to the problems of elastodynamics. The solution of the equation of motion (EOM) in combination with boundary conditions cannot in general be found in a closed analytical form. Therefore, a numerical solution form is sought where the mathematical model is transformed into a discrete approximate model. The most commonly used numerical methods are the finite element method (FEM) and the boundary element method (BEM). Several other approaches are available in low, high and medium frequency ranges. Those will be described in the following part of this paper.

2. Wave Approach
In the wave approach one considers travelling waves within the system caused by initial deformation due to loading. First the wave propagation in an unbounded homogeneous elastic medium will be considered. Then the problem of reflection and refraction at the discontinuity of the medium will be presented. Finally, we continue with the simplified theories of wave propagation in wave guides (structural elements) like rods, beams and plates which allow for analysis of engineering structures.

2.1. Wave Motion in an Unbounded Medium
The problem of wave motion in an unbounded medium has been presented among others by Achenbach,[1] and Graff, [20]. By solving the differential equation of motion (EOM) one obtains the wave equations. The EOM of a homogeneous, isotropic, linearly elastic body is described in terms of the displacement vector $\mathbf{u}(\mathbf{x}, t)$, where $\mathbf{x} = (x_1, x_2, x_3)$ denotes the coordinates of the point, as, [1]:

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla \nabla \mathbf{u} + \rho \mathbf{f} = \rho \dot{\mathbf{u}}$$

where $\lambda$ and $\mu$ are the Lamé constants. These equation represent Navier's equations. The use of Helmholtz’ decomposition of the displacement vector, $\mathbf{u}(\mathbf{x}, t)$ and unit mass loading vector $\mathbf{f}(\mathbf{x}, t)$ yields:

$$(2.1) \quad \mathbf{u} = \nabla \phi + \nabla \times \mathbf{\psi} \quad ; \quad \nabla \cdot \mathbf{\psi} = 0$$

$$(2.2) \quad \mathbf{f} = \nabla \mathbf{f} + \nabla \times \mathbf{F} \quad ; \quad \nabla \cdot \mathbf{F} = 0$$

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Helmholtz’ decomposition of the vector produces scalar \((\phi, f)\) and vector \((\psi, F)\) potentials. First consider the homogeneous equation, i.e. in the absence of body forces. Then by use of Helmholtz’ resolution equation (2.1) yields:

\[
\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla \nabla \mathbf{u} = \rho \ddot{\mathbf{u}} \tag{2.4}
\]

\[
\nabla \cdot \nabla \phi + \nabla \times \mathbf{\psi} + (\lambda + \mu) \nabla \nabla \cdot [\nabla \phi + \nabla \times \mathbf{\psi}] = \rho \frac{\partial^2}{\partial t^2} [\nabla \phi + \nabla \times \mathbf{\psi}] \tag{2.5}
\]

Using \(\nabla \cdot \nabla \phi = \nabla^2 \phi\) and \(\nabla \cdot \nabla \times \mathbf{\psi} = 0\) one may rearrange (2.5) into form:

\[
\nabla [(\lambda + 2\mu) \nabla^2 \phi - \rho \ddot{\phi}] + \nabla \times [\mu \nabla^2 \mathbf{\psi} - \rho \ddot{\mathbf{\psi}}] = 0
\]

which reduces to two wave equations:

\[
\nabla^2 \phi = \frac{1}{c_{L}^2} \ddot{\phi} \quad ; \quad c_{L}^2 = \frac{\lambda + 2\mu}{\rho} \tag{2.7}
\]

\[
\nabla^2 \psi = \frac{1}{c_{T}^2} \ddot{\psi} \quad ; \quad c_{T}^2 = \frac{\mu}{\rho} \tag{2.8}
\]

\(c_L\) is the propagation velocity of longitudinal waves (2.7), also called volumetric waves or primary (P) waves. \(c_T\) is the propagation velocity of shear waves (2.8), also called rotational, equi-voluminal, or secondary (S) waves. In the presence of body forces the inhomogeneous wave equations have form:

\[
\nabla^2 \phi - \frac{1}{c_{L}^2} \ddot{\phi} = -f \quad ; \quad \nabla^2 \psi - \frac{1}{c_{T}^2} \ddot{\psi} = -F
\]

The wave type depends on the type of excitation. The most straightforward type of excitation to analyse is the time dependent point force. In this case the solution has the polar symmetry, with the centre of the sphere being at the point of loading. This property allows for reducing the problem from a three-dimensional to a one-dimensional problem. The front of the wave generated by point load propagates away from the source. The solution to line load (in-plane and out-of-plane) is also possible to obtain, but it is more difficult than the one of polar symmetry.

2.1.1. **Plane Waves in a Homogeneous and Isotropically Elastic Material**

The type of waves, which are of general interest, are the plane waves. Those are the waves which are uniform in planes normal to propagation vector. Additionally we focus on harmonic waves, which can be described by the amplitude and harmonic function of time, i.e. \(\mathbf{u} = u_0 e^{i\omega t}, \mathbf{f} = f_0 e^{i\omega t}\). The plane wave propagating in the direction of the unit propagation vector \(\mathbf{p}\) is represented by [1]

\[
\mathbf{u} = \mathbf{f}(\mathbf{x} \cdot \mathbf{p} - ct) d
\]

The velocity of propagation \(c\), and the direction of motion \(d\) depend on the type of wave. By substituting the expression for the plane wave (2.10) into the homogeneous part of EOM (2.1) one obtains

\[
(\mu - \rho c^2) \mathbf{d} + (\lambda + \mu)(\mathbf{p} \cdot \mathbf{d}) \mathbf{p} = 0 \tag{2.11}
\]

This equation can be satisfied if either \(\mathbf{p} = \pm \mathbf{d}\) or if \(\mathbf{p} \cdot \mathbf{d} = 0\). If the direction of motion and the direction of propagation are parallel, i.e. \(\mathbf{d} = \pm \mathbf{p}\) the velocity of propagation is equal to \(c = c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}\) and it may be recognized as longitudinal, or P-waves.

If \(\mathbf{p}\) and \(\mathbf{d}\) are not parallel, they have to be perpendicular giving the vector product \(\mathbf{p} \cdot \mathbf{d} = 0\). Then the velocity of propagation equals \(c = c_T = \sqrt{\frac{\mu}{\rho}}\). This is the transverse or S-wave.

Plane harmonic waves are mostly used in the analysis of structural elements (wave guides). There it is assumed that the dimension of the cross-section is small enough to allow for propagation of plane waves only. Therefore, the plane waves will be of most interest in what follows.

2.2. **Reflection and Transmission of Waves at Discontinuity**

A discontinuity has a great impact on the propagation of the waves. The wave reaching the discontinuity is partly reflected and partly transmitted so that stresses and displacements are continuous on both sides of the discontinuity. The wave reaching the discontinuity is called incident. Depending on the type of the incident wave, type of discontinuity and the angle of incidence different phenomena may occur. Basically two possibilities exist for the type of discontinuity, i.e. the joined half-spaces, see figure 2.1. If both of
them are elastic media then both reflection and transmission will occur. If one of the media does not transmit mechanical waves, then only the reflection will take place.

2.2.1. Plane Wave Incident on the Joined Elastic Half-spaces

The wave reaching discontinuity is transmitted and reflected. The continuity conditions of displacements and stresses govern the amplitudes of reflected and transmitted waves. In the general case two reflected and two transmitted waves can occur. Figure 2.1 presents the propagation vectors $p$ and angles of incidence $\theta$ of:

- incident wave $p^{(0)}$, $\theta_0$,
- reflected longitudinal wave $p^{(1)}$, $\theta_1$,
- reflected transverse wave $p^{(2)}$, $\theta_2$,
- transmitted longitudinal wave $p^{(3)}$, $\theta_3$,
- transmitted transverse wave $p^{(4)}$, $\theta_4$.

![Figure 2.1](image)

*Figure 2.1. Reflection and transmission of the incident wave at the interface of two elastic media*

More detailed analysis of the reflection and transmission of waves at the interface of two elastic media may be found in [1, 20].

2.2.2. Wave Propagation in Wave-guides

For engineering use a simplified approach of analysis exists. By considering the differential equations of motion of particular elements, like:

- one-dimensional: thin rod, string, thin beam,
- two-dimensional: membrane, thin plate, shell,

one may define certain wave types, which are free to propagate in those structures. The equation of motion of a homogeneous elastic rod is given by [41]:

$$\frac{\partial}{\partial x} \left[ EA(x) \frac{\partial u(x, t)}{\partial x} \right] - \rho A(x) \frac{\partial^2 u(x, t)}{\partial t^2} = 0$$

(2.12)

where $x$ is the axial coordinate of the rod, $u$ is the axial displacement, $E$ is the elasticity modulus, and $A$ is the cross-sectional area. If the rod has constant cross-section $A$ then (2.12) reduces to:

$$\frac{\partial^2 u(x, t)}{\partial x^2} - \rho \frac{\partial^2 u(x, t)}{\partial t^2} = 0$$

(2.13)

or by using the standard wave equation:

$$\frac{\partial^2 u(x, t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u(x, t)}{\partial t^2} = 0$$

(2.14)
where \( c = \sqrt{E/\rho} \) is the wave velocity. The basic solution of one-dimensional wave equation can be written in the form:

\[
 u(x, t) = U^+(x - ct) + U^-(x + ct)
\]

The displacement propagating in the positive \( x \) direction is denoted \( U^+ \) and in the negative \( x \) direction by \( U^- \). In terms of harmonic components the solution of wave equation may be written as:

\[
 u(x, t) = \sum U_0^+ e^{-i(kx-\omega t)} + \sum U_0^- e^{-i(-kx-\omega t)}
\]

where the wave number \( k = \omega/c \) has been introduced, \( \omega \) and \( U_0 \) being, respectively, the angular frequency and amplitude of the harmonic component. We may focus on a harmonic component in further analysis, and use the Fourier series representation for other types of periodic motions if necessary.

From (2.16) one deduces that (in the absence of damping) the wave propagates towards \( \pm \infty \) without change of amplitude, wavelength, etc. This holds for an unbounded, homogeneous medium. At the discontinuity reflection/transmission takes place. The amplitudes of reflected/transmitted waves can be found by considering boundary conditions at the discontinuity.

As an example, let us consider two semi-infinite rods joined along their axes, see Fig. 2.2. The harmonic wave of amplitude \( U_I^+ \) travelling along the \( x \)-axis in rod \( I \) from \(-\infty \) reaches the discontinuity (i.e. change of material, cross-section, etc.). The wave transmitted to rod \( II \) has amplitude \( U_{II}^+ \). The reflected wave has the amplitude \( U_I^- \) (minus indicates that the reflected wave propagates towards \(-\infty \). The propagation velocity may be different in both rods, so may be the wavenumber. This is indicated by subscripts (\( I \) and \( II \)). The frequency of harmonic motions \( \omega \) is the same in both rods. The amplitude of transmitted/reflected waves is usually expressed in terms of incident wave amplitude and transmission (t)/reflection (r) coefficient, i.e.

\[
 U_{II}^+ = t U_I^+, \quad U_I^- = r U_I^+
\]

The reflection and transmission coefficients are to be found from boundary conditions, i.e. the equality of velocities and forces at both sides of the joint.

Detailed analysis of the wave propagation in rods may be found in [14, 15]. Table 2.1 presents the wave guides and the types of waves, which should be considered when analyzing the element. One has to keep in mind that the simplified theories of wave propagation are based on several assumptions concerning the proportions of the structure (length to height and width ratio for beams and rods, thickness to dimensions of the plate etc.). There are also certain conditions concerning wavelengths which have to be fulfilled, i.e. the waves have to be longer than the dimension of the structure which is withdrawn from the analysis. The detailed analysis of bent and coupled beams is presented in [23]. The approach of modelling the coupling as the mass element is presented, and the results are presented for bent beams as a function of a bent angle, and for three in-plane coupled beams. The tree-dimensional joint of beams was used in [10]
to present the energy flow analysis (EFA) method. A similar approach was presented in the case of rods with significant change in cross-section [46].

The analysis of plates by wave approach has been presented for example in [1, 14] and recently by Mace in [57, 37, 38].

2.2.3. Influence of Damping

Equation (2.1) may include damping forces as internal forces. But the modelling of damping is an open issue, since it is not defined similarly to elasticity modulus or mass density. There are different models of damping used in analysis of elastic materials. Often damping is introduced by setting the imaginary part of the elasticity modulus. When considering the propagating waves the complex elasticity modulus causes the amplitude of wave to decrease as it propagates out of the source. The wavenumber becomes complex, too, i.e. $k = \Re(k) + i \Im(k)$ Thus, in the presence of damping the additional waves, the so-called near field waves appear (they vanish as $e^{\Im(k)x}$ with increasing distance from the wave source).

It should be pointed out that the consideration of travelling wave behaviour at the structure discontinuity is probably the most efficient way of analyzing the coupled structures. It has been successfully used to analyze joints of different complexity. The solution is used in different techniques as a starting point to describe the power flow, or other quantities of interest.

The simplification in comparison with an explicit solution of EOM is mostly because one considers a set of wave components to describe the actual response to loading, which may have arbitrary shape. Therefore, for some types of loading, like harmonic point load, only one wave component is generated and the solution is exact.

3. Numerical Methods in Low Frequency Domain

In engineering practice the analytical solutions of the equations of motion (EOM) of the structure are hardly ever obtainable. Then the numerical methods have to be used. Analytical methods are often more accurate and more insightful when solutions are available, than numerical solutions. In general analytical and numerical methods are complementary and should both be used as appropriate. The various numerical methods such as e.g. the finite element method or the boundary element method are distinguished from each other by the mathematical formulation used and the approximations made. In the following only the general principle of FEM will be presented.
3.1. FEM Formulation

Several formulations lead to the basic multi-degree-of-freedom (MDOF) description of continuous structures. Among them there is the displacement-based method using the principle of virtual displacements. Let us consider the elastic body occupying the volume $V$. The boundary conditions are prescribed at the surface $S$ of the body in terms of surface displacements $u^S_t$ defined at the surface $S_t$ and the surface traction $f^S_t$ defined at the surface $S_t$. The body is subjected to concentrated loads $f^C_i$ at the position $x_i$ and external body forces $f^B$ (per unit volume). The latter one includes inertia forces.

The principle of virtual displacements states that the equilibrium of the body requires that the internal work (due to stresses $\sigma$) is equal to the total external virtual work, i.e. [3]

$$\int_V \varepsilon^T \sigma dV = \int_V \overline{\varepsilon}^T f^B dV + \sum_i \int_{S_i} \overline{\varepsilon}^T f^S_i dS + \sum_i \overline{\varepsilon}^T f^C_i \tag{3.1}$$

where the line over the symbol denotes virtual quantity. In (3.1) $\overline{\varepsilon}$ are the virtual displacements.

The FEM discretization consists of dividing the body into elementary volumes. The volumes are connected by nodal points at their boundaries. Partitioning into elements yields equation (3.1) to have the form:

$$\sum_m \int_{V^{(m)}} \varepsilon^{(m)T} \sigma^{(m)} dV^{(m)} = \sum_m \int_{V^{(m)}} \overline{\varepsilon}^{(m)T} f^{B^{(m)}} dV^{(m)}$$

$$+ \sum_m \int_{S^{(m)}_1, \ldots, S^{(m)}_q} \overline{\varepsilon}^{S^{(m)T}} f^{S^{(m)}} dS^{(m)} + \sum_i \overline{\varepsilon}^T f^C_i \tag{3.2}$$

The summation covers all elements of the volume, and the surfaces of the element $m$, which are the surface of the body, are denoted $S^{(m)}_1, \ldots, S^{(m)}_q$.

The displacement field of the single element 'm' is described as:

$$u^{(m)}(x, t) = \Psi^{(m)}(x)q(t) \tag{3.3}$$

where $\Psi^{(m)}$ is the displacement interpolation matrix with the superscript $(m)$ denoting the $m$th element; $q$ is a vector of coordinates, i.e. $q^T = [q_1, q_2, \ldots, q_n]$ where $n$ is the number of coordinates.

The element strains may be evaluated as

$$\varepsilon^{(m)}(x, t) = B^{(m)}(x)q(t) \tag{3.4}$$

where $B^{(m)}$ is the strain-displacement relation following the choice of $\Psi^{(m)}$. The stresses are related to strains and initial stresses, i.e.

$$\sigma^{(m)}(x, t) = C^{(m)} \varepsilon^{(m)} + \sigma^I^{(m)} \tag{3.5}$$

where $C^{(m)}$ is the elasticity matrix of element $m$, $\sigma^I$ are the initial stresses.

Using the same discretization as in (3.3, 3.4) for the virtual element displacements $\overline{u}^{(m)}$ and strains $\overline{\varepsilon}^{(m)}$, i.e.

$$\overline{\varepsilon}^{(m)}(x, t) = \overline{\Psi}^{(m)}(x)\overline{q} \tag{3.6}$$

$$\overline{\varepsilon}^{(m)}(x, t) = B^{(m)}(x)\overline{q} \tag{3.7}$$

and substituting into (3.2) one obtains:

$$\overline{q}^T \left[ \sum_m \int_{V^{(m)}} B^{(m)T} C^{(m)} B^{(m)} dV^{(m)} \right] q = \overline{q}^T \left[ \sum_m \int_{V^{(m)}} \overline{\Psi}^{(m)T} f^{B^{(m)}} dV^{(m)} \right]$$

$$+ \left\{ \sum_m \int_{S^{(m)}_1, \ldots, S^{(m)}_q} \overline{\Psi}^{S^{(m)T}} f^{S^{(m)}} dS^{(m)} \right\} - \left\{ \sum_m \int_{V^{(m)}} B^{(m)T} \sigma^I^{(m)} dV^{(m)} \right\} + f^C \tag{3.8}$$

This can be simplified by defining:
• the stiffness matrix:

\[
K = \sum_{m} \int_{V(m)} B^{(m)T} C^{(m)} B^{(m)} dV(m);
\]

(3.9)

• the element body forces vector:

\[
f_B = \sum_{m} \int_{V(m)} \Psi^{(m)T} f^{B(m)} dV(m);
\]

(3.10)

• the element surface forces vector:

\[
f_S = \sum_{m} \int_{S^{(m)}} \Psi^{S(m)T} f^{S(m)} dS^{(m)};
\]

(3.11)

• the element initial stresses:

\[
f_I = \sum_{m} \int_{V(m)} B^{(m)T} \sigma^{I(m)} dV(m).
\]

(3.12)

One may combine (3.10 - 3.12) and nodal concentrated loads \(f_C\) into the load vector \(f\):

\[
f = f_B + f_S - f_I + f_C
\]

(3.13)

and by use of the Gauss elimination one may write (3.8) as:

\[
Kq = f
\]

(3.14)

### 3.2. FEM Applied to Dynamic Analysis in Time Domain

In the dynamic analysis the inertia forces can be included in the body force vector. Then equation (3.10) must be rewritten as:

\[
F_B = \sum_{m} \int_{V(m)} \Psi^{(m)T} \left[ f^{B(m)} - \rho^{(m)} \Psi^{(m)} \ddot{q} \right] dV(m);
\]

(3.15)

where, using d’Alembert’s principle, the inertia forces are proportional to the acceleration \(\ddot{u}\) and the mass density \(\rho^{(m)}\) of the element. By substituting:

\[
M = \sum_{m} \int_{V(m)} \rho^{(m)} \Psi^{(m)T} \Psi^{(m)} dV(m)
\]

(3.16)

one obtains the equilibrium equation in the form:

\[
M\ddot{q} + Kq = f
\]

(3.17)

In a similar way the velocity-proportional damping effect may be included in the analysis, i.e.:

\[
M\ddot{q} + C\dot{q} + Kq = f
\]

(3.18)

There are at least three issues influencing the applicability of FEM to dynamic analysis:

• increasing size of the problem
• application of the approximate theory in high frequency analysis,
• deterministic description of the uncertainties of reality.

The first one is due to the fact that there must be at least 5-10 nodes within a wavelength to represent the motion of the structure. The wavelengths of the displacements decrease with increasing frequency, so in order to keep this dependence the size of elements must be decreased. As a result the number of DOF becomes huge and the calculations become impossible to carry out. The idea of sub-structuring was the natural method of handling problems when the complexity increased, and the models become difficult to treat by numerical methods. The analysis of components is easier to handle in the numerical calculation. This is the big advantage of the substructure method. The problem of coupling appears, however, and it is seen to be the main barrier in using substructure method.

In the natural way the coupling properties follow the boundary conditions between structures. The definition of boundary conditions depends on the method used in substructure analysis, i.e. the boundary conditions should involve the same variables as the model. As it was mentioned before, the widely used method of dealing with the coupling is by considering it in terms of travelling waves.
The FEM is derived as a discretization of some approximate continuum mechanical theory such as the Bernoulli-Euler beam theory or the thin plate theory. In this case it is well known that a characteristic wavelength of the considered motions should be well above say 5-10 element widths if sufficient agreement between numerical and analytical solutions is to be expected. Secondly, as the wavelength of the mode shape becomes comparable to the plate thickness or the beam height the approximate continuum mechanics becomes increasingly useless.

The latter problem deals with uncertainties in the manufacturing, which mostly influences the high-frequency behaviour of the structure. It should be noticed that FEM used to solve unbounded problems has to be combined with methods, which represent the influence of the infinite medium, not considered here.

### 3.3. FEM Applied to Dynamic Analysis in Frequency Domain

By taking Fourier transform of equation (3.18) one obtains

$$ \left[ -\omega^2 M + i\omega C + K \right] Q = DQ = F $$

where $Q(i\omega)$ and $F(i\omega)$ are the Fourier transforms of $q(t)$ and $f(t)$, respectively. In equation (3.19) the dynamic stiffness matrix $D$ is introduced [34]. The dynamic stiffness matrix is one of the frequency response functions (FRF) describing the response of the structure to the harmonic loading as a function of frequency. Table 3.1 summarizes frequently used frequency response functions.

<table>
<thead>
<tr>
<th>FRF</th>
<th>quantity / quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>dynamic stiffness matrix $D$</td>
<td>force / displacement</td>
</tr>
<tr>
<td>impedance $Z$</td>
<td>force / velocity</td>
</tr>
<tr>
<td>mobility $Y$</td>
<td>velocity / force</td>
</tr>
</tbody>
</table>

**Table 3.1. Frequency response functions**

### 3.4. Power Flow FEM

One of the possible methods of reducing the FEM model is by considering the energy flow in the structure [42]. This approach has the advantage of SEA, see section 4.3, which is the energy description. The main difference is that the energy dissipation and conduction within the subsystem are taken into account instead of the mean value (as in SEA). It is assumed that the total vibrational energy $w$ of unit volume $V$ is twice the kinetic energy, thus given by:

$$ w = \rho \bar{v}^2 $$

where $\bar{v}$ is the time averaged particle velocity in the frequency band $\Delta f$, and $\rho$ is the mass per unit volume. The conservation of energy yields the power balance between the variation of energy density $\partial w / \partial t$, the energy flux density crossing the boundary $\bar{q}$ and the power $p_{diss}$ dissipated by unit volume:

$$ \frac{\partial w}{\partial t} = -\nabla \cdot \bar{q} - p_{diss} $$

The power dissipated by unit volume is assumed to be proportional to the energy of the volume, i.e. $p_{diss} = \delta w$, where $\delta$ is energy dissipation coefficient. The energy flux is assumed to have the form:

$$ \bar{q} = -\lambda \nabla (\kappa w) $$

$\lambda$ and $\kappa$ are the vibration conduction coefficients assumed to be scalars. With the simplification mentioned, equation (3.21) obtains the form

$$ \frac{\partial w}{\partial t} = \nabla \cdot \lambda \nabla (\kappa w) - \delta w $$

which is analogous to the heat conduction equation. By introducing $W = \kappa w$ one obtains

$$ \frac{\partial W}{\partial t} = \kappa \nabla \cdot \lambda \nabla W - \delta w $$

which is the heat transfer equation in the standard form and may be simply modelled by finite element formulation. The solution of the steady-state or transient problem may be obtained as soon as the coefficients $\delta, \kappa$ and $\lambda$ are defined. In [42] the derivation of those coefficients, based on wave approach is shown, for the case of coupled beams. The basic assumption of analogy between the distribution of
thermal and mechanical energy was criticized in [7]. It was shown that for the general elastic medium in the frequency domain the analysis should be based on Navier’s equation (2.1). The power balance equation for the undamped structure may not be written as in equation (3.24) [7]. The more complicated form is required and it is not more convenient than the original displacement formulation. This consideration shows, however, the inaccuracy in thermal analogy in comparison with an exact solution. Therefore, another approach was proposed in order to simplify the energy flow relation. In [47] the authors propose to consider the envelope of the displacement only, not the displacements by themselves. This leads to envelope energy model (EEM).

4. Numerical Methods in High Frequency Domain

By high frequency region one understands the region where the statistical methods of vibration analysis give reliable predictions. This requires that the response of the structure has a flat spectrum, i.e. the resonant peaks are not visible due to modal overlap in the system. This region is also characterized by high uncertainty in modal parameters, which causes deterministic methods, like FEM to be unreliable.

4.1. Mean Value Method

The analysis of vibrational behaviour of the structures has shown that at particular frequencies the structural response may be governed by the response of one member. Moreover the resonant peaks and anti-resonant thoughts of the response function oscillate around the response of the similar system of infinite dimensions. The argument used is that the resonance appears when the wave field caused by reflections from boundaries together with the original one are in phase, so there is an amplification of the response. In anti-resonance the two wave fields are out of phase, so the vector sum becomes the difference. In the absence of boundaries (the infinite structure) the response is smooth and it indicates the mean response of the element of the structure. In addition one may calculate the response function for simple structural members analytically, and this may be used as the indicator of the behaviour of the structure. The method has been presented by Skudrzyk in [49] and recently by Hugin, [26].

4.2. Envelope Analysis

More recently several authors proposed to include the envelope of FRF as being of great importance, and have shown the maximum and minimum value to have similar properties as mean in terms of describing average properties of the set. Those considerations may be found in [29, 47, 8]. First the energy envelope method (EEM) has been introduced. The EEM was later developed into envelope-phase energy model (EPHEM). Finally the complex envelope displacement analysis (CEDA) has been introduced [8] which is the quasi-static approach to vibrations.

4.2.1. Complex Envelope Displacements Analysis (CEDA)

CEDA relies on a suitable variable transformation, whose aim is to reduce the modelling expenses, but also to enable modelling of coupling between structures in the similar manner as the structures themselves. This is an extension of EEM and EPHEM where the suitable definition of coupling was not straightforward. The method was used sufficiently in [8] for the pair of coupled beams and the results were compared with the exact solution. The results show good agreement, even though the number of samples in the exact solution exceeds ten times the number of samples in the approximate solution.

4.2.2. Basic Relation of CEDA

The equation of motion for a one-dimensional undamped structure may be written in the form

\[ L[u(x)] + m\omega_0^2 u(x) = f(x) \]

where \( L \) is the differential operator, \( u(x) \) is the displacement vector and \( f(x) \) is the external loading vector. The loading and response are assumed to be harmonic with frequency \( \omega_0 \) in the steady state conditions. The complex envelope of displacement \( \tilde{u} \) can be introduced through the action of an envelope operator \( E \) on the displacement vector \( u \)

\[ E(\cdot) = [I(\cdot) + i\, H(\cdot)]e^{-i\omega_0 x} \]

where \( H \) signifies the Hilbert transformation [55], \( I \) is the identity transformation, \( k_0 = \frac{\omega}{c_0} \) is the wave number, corresponding to the frequency of excitation, \( c_0 \) is the phase wave speed in the system. The use
of envelope operator in equation (4.1) leads to
\begin{equation}
E\{L[u(x)]\} + m\omega_0^2 E[u(x)] = E\{f(x)\} = \tilde{f}(x)
\end{equation}
and by substituting the complex envelope of displacement \( \tilde{u} \) instead of \( u \) one obtains
\begin{equation}
ELE^{-1}\{\tilde{u}(x)\} + m\omega_0^2 \tilde{u}(x) = \tilde{f}(x)
\end{equation}
By introducing the complex envelope operator \( \tilde{L} = ELE^{-1} \) one obtains the complex envelope equation in the form
\begin{equation}
\tilde{L}[\tilde{u}] + m\omega_0^2 \tilde{u} = \tilde{f}
\end{equation}
The method allows for inverse transform, closer reported in [8].

The operators for structural problems like an Euler-Bernoulli or a Timoshenko beam were presented. In the analysis the simplification proposed by Langley [28], i.e. the extraction of near and far solution is used. The comparison between computation of two coupled beam interactions both by the CEDA method and the direct displacement analysis is performed. The CEDA solution converges well with the exact one, but is more efficient in the meaning of computation time. The problem of radiation from the structure is reported as an object of further consideration as a necessary part of vibroacoustic analysis. The method appears to be the promising one in the analysis of dynamic behaviour of structures in the mid frequency range.

4.3. Statistical Energy Analysis - SEA

For the high frequency response of complicated structures, especially in the building acoustics the frequently used method is the SEA. The origins of SEA grow out from the limitations of deterministic models and analysis. The first one is the simplification of the model in case of high frequency analysis and uncertainty associated with such simplification. The second one is the sensitivity of modal resonance frequencies to small changes in structural details like boundary conditions, damping. The next one is the uncertainty in load modelling, which in reality is not deterministic. These are the reasons why a global description of system properties was developed, which allows for the global response for excitation to be found. This description is statistical, taken over the ensemble and in this meaning the response is uniform over the ensemble. The properties are also averaged spatially and over frequency band to simplify the model. In fact the SEA approach deals with the frequency-averaged value of energy response function of subsystems and relates the time-averaged power flow between subsystems to their steady-state energies.

4.3.1. Conventional SEA

The SEA basic equation is the power balances for subsystems in the form
\begin{equation}
\Pi_i^{in} + \sum_k \Pi_{k\rightarrow i} = \Pi_i^{diss} + \sum_j \Pi_{i\rightarrow j}
\end{equation}
Where \( \Pi_i^{in} \) is the input power from external loading to the system \( i \), \( \Pi_{k\rightarrow i} \) is the power transmitted from subsystem \( k \) to subsystem \( i \), \( \Pi_i^{diss} \) is the dissipated power (due to internal damping, radiation into surrounding, etc.), and \( \Pi_{i\rightarrow j} \) is the transmitted power from subsystem \( i \) to \( j \), see fig. 3.

It is assumed that during analysis the power input is known. The dissipated power is assumed to be proportional to the subsystem energy \( E \). The transmission of the power requires, however, deeper consideration. In the modal approach, subsystems are treated as sets of modes. The energy exchange between subsystems follows the interaction between modes belonging to different subsystems, and may therefore be expressed based on relations for two coupled oscillators. The original relation given by Scharton and Lyon is valid for lightly damped conservatively coupled oscillators subjected to white noise excitation. It governs the exchange of energy between oscillators, i.e.
\begin{equation}
E[\Pi_{12}] = g_{12}(E[E_1] - E[E_2])
\end{equation}
The analysis of multi-modal systems follows (4.7). If one assumes that:
- acting forces are uncorrelated;
- the coupling is 'weak', so the interaction between two modes from the same set is negligible compared with interaction of two modes from different sets;
- energy is shared equally among modes in the frequency band of interest;
then the net power flow may be written in the form

\[ E[\Pi_{12}] = \omega(\eta_{12} E[\Pi_1] - \eta_{21} E[\Pi_2]) \]

\( \eta_{ij} \) is the coupling loss factor. There are several definition when the coupling is 'weak'. The subject of strong coupling was also studied, and will be mentioned later on. The modal approach is used in case of vibroacoustic interaction between structure and fluid. It is rarely used to predict structural vibration, though there are several examples of analysis of rods, beams, rectangular plates. It is however more common to analyse the transfer of energy in terms of traveling waves, the so-called wave approach. The reflection of the waves at the junction and their transmission through is considered. The main assumption in wave approach is that the outgoing and reflected waves are uncorrelated. This is not true in case of resonance which may appear for low modal overlap. That is why the wave approach is appropriate for high frequency analysis, where the energy-response function has no resonant peaks. When both damping loss factors (DLF) and coupling loss factors (CLF) are given, then the simple calculation allows predictions of the system response. Also further analysis is available.

The general advantages of SEA are:

- low number of variables is necessary to describe the system in the high frequency range,
- the results are insensitive to small changes in properties - the average over an assemble is taken in a simple way,
- calculations are relatively simple.

The general disadvantages of SEA are:

- information about spatial distribution of variables is lost,
- there no no methods to check reliability of SEA predictions, i.e. non-resonant interaction or weak coupling,
- the SEA factors, i.e. CLF or DLF are difficult to obtain for more complicated structures.

4.3.2. **Necessary Development of SEA**

There are several fields where the SEA predictions are incorrect, and those fields are of main interest for researchers. The most important ones are:

- strong coupling, the definition and influence on the predictions,
- non-conservative coupling,
- low frequency analysis, the measure for lowest frequency,
• periodic or semi-periodic structures.

Some of those items were reported, and studied, i.e.:

• calculating CLF’s and DLF’s based on ‘numerical experiments’ using FEM, [22, 48, 13, 53, 19],
• the method of prediction of the minimum frequency SEA model is proposed in order to ensure the reliability level, [39],
• the measure of coupling weakness has been proposed in [27], another one in [17],
• resonant behaviour of the coupling elements, not suitable for SEA modelling, was presented and discussed in reference [2],
• the problem of periodic and semi-periodic structures has been studied in [30, 31, 33, 32, 35, 56],
• the non-conservatively coupled structures were considered in [36, 16, 9] (oscillators), [54, 5, 6],
• the wave approach to derive the CLF for coupled beams in low frequencies is presented in [24, 25].

5. Numerical Methods in Medium Frequency Domain

The definition of the mid frequency range is difficult, and not straightforward. There are no exact limits of that frequency, it is only described by system properties. According to definitions used in [43, 18] and others the medium frequency band is the one where the conventional methods (like FEM, substructure approach) are not appropriate but the SEA assumptions are not yet fulfilled. The other definition says that the medium frequency band for the structure is the one, where some of the members may already be treated by SEA, where the other one must be analyzed by other methods. To overcome the difficulties in handling the analysis in mid frequency band, several new approaches have been proposed:

• combined FEM and SEA analysis by use of substructure approach, and choice of method for each member,
• combined FEM and SEA, but the same member is modelled by both methods,
• combined FEM and integral description,
• combined FEM and analytical solution coupled by Lagrange multipliers,
• parameter-based statistical energy method.

5.1. Iterative FEM + SEA

The first method has been proposed by Wilson, [58]. It relies on substructure approach. The coupling between different descriptions is not explicit, the output from FEM analysis creates the input to SEA. The SEA calculations are then performed and the back loading calculated. The calculations are then repeated, so the method is iterative, and the power balance is calculated until equilibrium is reached.

5.2. Local and Global Description

The second approach has been proposed by Langley, [34]. Equation (3.3) is modified by choosing two sets of admissible functions, referred to as “local” and ”global”. Then the decomposition in terms of ”local” and ”global” functions is performed, i.e.

\[ u(x, t) = \sum_{j=1}^{N_2} q^g_j(t)\Psi^g_j(x) + \sum_{j=1}^{N_1} q^l_j(t)\Psi^l_j(x) = [\Psi^g \quad \Psi^l] \begin{bmatrix} q^g \\ q^l \end{bmatrix} \]

The definition of ”global” and ”local” is not precise, but it is assumed that for build-up structure it is possible to separate the local and global behaviour. The main idea in the hybrid approach is that the local motions are analyzed in statistical manner, i.e. in terms of statistical energy analysis, while the global motions are analyzed by FEM. This approach may be seen as a kind of fuzzy approach [50, 51, 52], where the global functions describe the structure and the local functions describe the attachments. The equation of motion (3.19) may be partitioned in terms of decomposition performed in (5.1) as follows:

\[ \begin{bmatrix} D^{gg} & D^{gl} \\
D^{lg} & D^{ll} \end{bmatrix} \begin{bmatrix} Q^g \\ Q^l \end{bmatrix} = \begin{bmatrix} F^g \\ F^l \end{bmatrix} \]

The global coordinates \( q^g \) can be selected as modal coordinates, and the matrix \( D^{gg} \) is thus diagonal (the necessary assumption is the one of diagonal damping matrix) with the \( nn \) entry in the form:

\[ (D^{gg})_{nn} = \omega_n^2(1 + \eta_n) - \omega^2 \]
with $\omega_n$ being the natural frequency of the $n$th global mode with the damping ratio $\eta_n$. It was shown in [34] that the coupling between globally and locally admissible functions may be approximated by:

\[ D_{gl} \approx -\omega^2 M_{gl} \]

where

\[ (M_{gl})_{mn} = \sum_{r=1}^{N_r} \int_{V_r} \rho_r(x) \psi_{gr}^T(x) \psi_{ir}(x) dx \]

Equation 5.2 may be rewritten into the form

\[ (D_{gg} - D_{gl} D_{ll}^{-1} D_{gll}^T) Q^g = F^g - D_{gl} D_{ll}^{-1} F^l \]

\[ D_{ll} Q^l = F^l - D_{gl}^T Q^g \]

Equation (5.6) describes the global motions, and (5.7) the local behaviour. The mixed terms are setup and the loading from local terms on global one is approximated, and finally, the equation for global functions is found. As mentioned before the local equations of motion are solved in terms of SEA. A more detailed presentation of the method is given in [34].

5.3. FEM + Integral Description

Another kind of hybrid analysis method has been proposed in [18]. The low frequency behaviour, where the properties depend upon structural details is modelled by FEM. The high frequency response is explained by use of asymptotic integral estimates. It is assumed that the high frequency vibrations are localized, and the substructure approach is used. Moreover, by use of this localization assumption the substructures are coupled only by low frequency response, not by the latter one. The integral approach allows for calculation of response envelopes, and should therefore be used above the frequencies of reliable FEM analysis.

5.4. FEM + Analytical Solution

Semi-analytical methods can also be used in medium frequency band. The idea of the method is to combine the analytical description with the FEM. The application of the method to model the connection of beams has been presented in [21], see fig.5.1. The coupling between the analytically described and part and the elastic finite element is described in terms of Lagrange multiplier.

![Figure 5.1. Connected FEM and analytical description of beams joint.](image)
5.5. Parameter-based Statistical Energy Method

Recently the modification to SEA has been proposed in [45, 11]. This approach aims at overcoming the SEA assumption concerning the distribution of natural frequencies. In SEA the appearance of natural frequencies is assumed to be constant for a subsystem within the frequency band. This assumption allows use of the single measure - modal density, instead of separate frequencies.

The Parameter-based Statistical Energy Method (PSEM) uses the probability density function (pdf) of natural mode appearance, instead of modal density. Thanks to this change the prediction includes the resonant behaviour in low frequencies (or low modal overlap). The pdf is calculated by considering the variation of parameters (uncertainties) in model description. The detailed formulation of the method and application to rods connected by springs have been presented in [45, 11]. The results show better agreement with the exact solution than SEA in case of low modal overlap. When modal overlap becomes high, both methods converge to the exact solution.

A similar approach has been proposed in [40]. The Statistical Modal Energy Distribution Analysis presented in this work also focuses on the distribution of natural frequencies, but in terms of functional approximation. Both methods seem to improve the SEA predictions in frequencies where the modal overlap is low.

5.6. Structural Fuzzy

The modelling of structures often face the problem how to deal with small attachments. In airplanes, submarines or ships the on-board equipment is a good example of a structure with uncertain properties. The position, mass, damping of this equipment are not known, but even if so it would not be possible to model it because of the huge number of required degrees of freedom. Simple treatment of these attachments would be by assuming that it acts as an added mass, but the experimental results [51] show that is not the case. The attachments act both added mass and damping. In addition the attached mass effect may be both positive and negative, and it is frequency dependent. The damping properties of the attachments mostly depend on their mass, and even if the attachments are lightly damped themselves, they still damp the vibrations of the main structure.

Most of the derivations, like [44, 50], deal with structures loaded by external fluid. In what follows this factor will be omitted.

By master structure one "designates the part of the mechanical system, which is accessible to conventional modelling, i.e. the system whose mechanical properties, geometry, boundary conditions and excitations are known with sufficient accuracy". In contrast the fuzzy structure or structural complexity is not accessible to conventional modelling. The possible reasons why the probabilistic modelling may be required are as follows:

- High number of internal equipment with uncertain properties. In this case, even if the properties would be known the quantity of required additional DOFs prevent deterministic modelling.
- Elements, which are omitted in modelling the master structure, because their resonance appears in medium frequency range and they are therefore assumed to have uncertain influence on the response.

The basic model of the structure with fuzzy attachments may be that of a rectangular plate with attached small oscillators. According to the literature the plate will be designed as a master structure and the attachments - fuzzy structure, or simply fuzzy. The out-of-plane motion of the plate denoted \( u_3(x_1, x_2) \) is governed by:

\[
(5.8) \quad B_{pl} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \frac{\partial^2 u_3}{\partial t^2} = m_{pl} \frac{\partial^2 u_3}{\partial t^2} + w - \sum_{n=1}^{N} F_n(t) \delta(x_1 - x_{1n}) \delta(x_2 - x_{2n})
\]

where \( B_{pl} \) is the bending stiffness of the plate, \( m_{pl} \) is the mass of the plate, \( m_n \) is the mass of the \( n \)th attachment and \( F_n \) is the force acting on the plate and caused by this attachment. The Dirac delta function \( \delta(x_1 - x_{1n}) \delta(x_2 - x_{2n}) \) has been used to indicate the \( n \)th attachment is connected with the structure at point \((x_{1n}, x_{2n})\). Each attachment is treated as a simple mass-spring system, \( M_n \) and \( K_n \), respectively. Instead of stiffness one may alternatively use natural frequency and replace \( K_n \) by \( \Omega_n M_n \). Damping is introduced as the fraction of critical damping, i.e. \( 2\zeta_n M_n \Omega_n \). Denoting the displacement of the mass by
\( z_n(t) \) one may write the equation of motion of the attachment as:

\[
M_n \frac{d^2 z_n}{dt^2} + 2 \zeta_n M_n \Omega_n \left( \frac{dz_n}{dt} - \frac{\partial u_3}{\partial t} |_{x_1=x_2} \right) + M_n \Omega_n^2 (z_n - u_3 |_{x_1=x_2}) = 0
\]

By using finite element discretization one may transform equation (5.9) into a standard form of equation (3.19), i.e.:

\[
[D_{\text{mast}} + D_{fuz}] Q = F
\]

The properties of fuzzy are stored in a frequency response function \( D_{fuz} \). The theory of structural fuzzy aims at approximating this term by considering statistical properties of the fuzzy.

The results presented by Soize, [51] for the beam and attached fuzzy substructures show the damping effect that the attached elements have on the vibrations of the master structure. The solution obtained by the fuzzy structure theory shows good agreement with the results of the numerical simulations of the fuzzy structure.

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**References**


Sohngaardsholmsvej 57, DK-9000 Aalborg, Denmark

E-mail address: ifmw@civil.au.dk