Energy Flow in Plate Assembles by Hierarchical Version of Finite Element Method

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Abstract

The dynamic analysis of structures in medium and high frequencies are usually focused on frequency and spatial averages of energy of components, and not the displacement/velocity fields. This is especially true for structure-borne noise modelling. For the analysis of complicated structures the finite element method has been used to study the energy flow. The finite element method proved its usefulness despite the computational expense.Therefore studies have been conducted in order to simplify and reduce the computations required. Among others, the use of hierarchical version of finite element method has been proposed.In this paper a modified hierarchical version of finite element method is used for modelling of energy flow in plate assembles. The formulation includes description of in-plane forces so that plates lying in different planes can be modelled. Two examples considered are: L-corner of two rectangular plates an a I-shaped plate girder made of five plates. Energy distribution among plates due to harmonic load is studied and the comparison of performance between the hierarchical and standard finite element formulation is presented.

Key words: finite element method, h-version, energy flow, thin plate, hierarchical fem

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1 Introduction

The prediction of transmission paths of vibrations is of importance in, among others, structural, automotive, marine and aviation engineering. The interest lies in assessment of energy flow within, often complicated structure at different frequencies, depending on applications. Two main methods are used generally to predict the vibrations of structures, namely the finite element method (FEM) and statistical energy analysis (SEA). There are basic differences between these two approaches to vibrations. The finite element method is used when the wavelengths of elastic deformation are long. The statistical energy analysis is used at frequencies where the modes overlap, for diffuse wave fields generated in the structure by multiple reflections of waves with short wavelengths.

There is an extensive effort in extending the possibility of FEM modelling into frequencies where it is too expensive computationally at present. One possibility is the use of description of the displacement field that converges better than the usual polynomials used in h-version of FEM (where the convergence is achieved by decreasing elements size), namely to use the hierarchical or p-version of FEM. Another one is the use of spatially averaged energy of subsystems as variables, instead of field variables. Both methods will be combined in this presentation.

Previous studies on hierarchical formulation of the plate problem in [1–5] focus on the case of only one plate, or plates laying in one plane. The comparison of natural frequencies of rectangular plates with ten different combinations of boundary conditions resulting from hierarchical FEM, the analytical solutions and the h-version of FEM [1, 6] shows the computational advantages of the hierarchical formulation in terms of convergence rate. Different sets of shape functions presented in literature include polynomial set of shape functions [2], the set of products of trigonometric functions [2, 3] or the combination of trigonometric functions and polynomials [4, 5].

Previous studies on the use of energy flow models from finite element analysis can be divided in two groups. The use of response variables calculated by FEM to define energy and its spatial / time averages was presented in [7–9]. The energy information can than be used to calculate the energy ratios or estimate coupling loss factors. The calculations can be repeated easily for different set of boundary conditions or modified geometry to account for uncertainties. Other studies, like [10–12] proposed the use of modal information of subsystems resulting from finite element modelling in calculating coupling loss factors (energy influence coefficients). In this approach the response is not calculated by finite element approximation. This method, however, requires that the uncertainties of the structure are transformed to uncertainties of modal parameters. This can be achieved by assuming distribution of natural frequencies and mode shapes or by resolving eigenvalue problem for different properties. The first one is difficult to accomplish in the case of uncertain boundary conditions, the second one require eigenvalue analysis of different ensemble members. For these two reasons the modal decomposition will not be used in this paper.

In this paper the combination of polynomials and products of trigonometric functions are used. The products of trigonometric functions defined in [2] are used for the shape functions of order 9 and above. For the first 8 shape functions the third order polynomials were used instead. Shape functions based on polynomials are also used for in-plane deformation, so that the continuity between plates laying in perpendicular planes is fulfilled along common edge.

The computational advantage of the presented method can be seen in two stages. Firstly, the faster convergence of the hierarchical FEM and thus the reduction of degrees of freedom for the primary variables. Secondly, the use of spatial average kinetic energy as the secondary variable for each homogeneous subsystem, which reduces the storage requirements. This is extremely important in the case of repeated calculations, for example in the case of averaging over excitation points.

2 Hierarchical Finite Element Method

This part presents the basics of the hierarchical FEM applied to plate problem. As in the h-version the domain is divided into elements. Natural co-ordinate system is defined for each element, with the co-ordinates ξ_1 and ξ_2 being in the plane of element. The origin is placed in the center of element, and the co-ordinates are normalized with respect to element geometry, i.e.

$$\boldsymbol{\xi}^e = [\xi_1^e, \xi_2^e]; \qquad \xi_i^e \in (-1, 1). \tag{1}$$

The displacement field $\boldsymbol{\upsilon^e}(\boldsymbol{\xi^e}, t)$ within each element (denoted by superscript 'e') is interpolated in terms of the shape functions $\boldsymbol{\Psi}(\boldsymbol{\xi}^e)$, defined in local co-ordinate system $\boldsymbol{\xi^e}$ and corresponding time dependent variables $q^e(t)$, i.e.

$$\boldsymbol{\upsilon}^e(\boldsymbol{\xi}^e, t) = \sum_n q_n^e(t)\boldsymbol{\Psi}_n^e(\boldsymbol{\xi}^e) \tag{2}$$

where the summation covers number of shape functions that is required to approximate the displacement field sufficiently. The element displacement field $\boldsymbol{\upsilon}^e(\boldsymbol{\xi}^e, t)$ can be divided into the in-plane displacement field: $\upsilon_1^e(\xi_1^e, \xi_2^e, t)$ and $\upsilon_2^e(\xi_1^e, \xi_2^e, t)$; and the out-of-plane displacement field: $\upsilon_3^e(\xi_1^e, \xi_2^e, t)$. Because the wavelengths of the in-plane deformation are much longer than the one of the out-of-plane deformation at the same frequency it is assumed here that the in-plane deformation can be approximated well without hierarchical formulation, just using the third-order polynomials. This assumption is not necessary, but it will simplify the presentation of the method.

The displacement field of the entire structure is given by the displacement fields of elements transformed into global co-ordinate system, i.e.

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{e=1}^{N_{el}} \boldsymbol{T}^e \boldsymbol{v}^e(\boldsymbol{\xi}^e,t) \tag{3}$$

where N_{el} denotes the total number of elements in the system; $\boldsymbol{T}^e$ denotes transformation matrix between the element co-ordinate system and the global co-ordinate system. The shape functions used to approximate the out-of-plane displacement field are defined as a product of two one-dimensional shape functions [2], i.e.

$$\boldsymbol{\Psi}_{L(k-1)+l}(\boldsymbol{\xi}^e) = \psi_k(\xi_1^e)\,\psi_l(\xi_2^e) \qquad k = 1\ldots K^e \;\; ; \;\; l = 1\ldots L^e \tag{4}$$

where K^e, L^e signify the highest interpolation function in element e in each of directions which can be chosen from the convergence criteria for each direction separately. The one-dimensional shape functions are defined in following manner:

$$\begin{aligned}
&\psi_1(\xi) = \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3; \qquad \psi_2(\xi) = \frac{1}{8} - \frac{1}{8}\xi - \frac{1}{8}\xi^2 + \frac{1}{8}\xi^3 \\
&\psi_3(\xi) = \frac{1}{2} + \frac{3}{4}\xi - \frac{1}{4}\xi^3; \qquad \psi_4(\xi) = -\frac{1}{8} - \frac{1}{8}\xi + \frac{1}{8}\xi^2 + \frac{1}{8}\xi^3 \\
&\psi_{n+4}(\xi) = \sin\left[\frac{n\pi}{2}(\xi+1)\right]\sin\left[\frac{\pi}{2}(\xi+1)\right].
\end{aligned} \tag{5}$$

The continuity conditions between subsequent elements have to be applied additionally to ensure the continuity of displacement and slope along elements edges.

2.1 Calculation of Stiffness and Mass Matrices

The potential and kinetic energy of the out-of-plane motions of thin plate assuming small displacements (Kirchhoff plate theory) may be written as [2]:

$$U = \frac{1}{2}\frac{4Db}{a^3}\int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{\partial^2 \upsilon_3}{\partial \xi_1^2}\right)^2 + \frac{1}{(b/a)^4}\left(\frac{\partial^2 \upsilon_3}{\partial \xi_2^2}\right)^2 \right.$$
$$\left. + \frac{2\nu}{(b/a)^2}\left(\frac{\partial^2 \upsilon_3}{\partial \xi_1^2}\right)\left(\frac{\partial^2 \upsilon_3}{\partial \xi_2^2}\right) + \frac{2(1-\nu)}{(b/a)^2}\left(\frac{\partial^2 \upsilon_3}{\partial \xi_1 \partial \xi_2}\right)^2\right] d\xi_1 d\xi_2 \tag{6}$$

$$T = \frac{1}{2}\frac{\rho h a b}{4}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial \upsilon_3}{\partial t}\right)^2 d\xi_1 d\xi_2 \tag{7}$$

where D is the plate bending stiffness, $D = Eh^3/12(1-\mu)$, ρ is the mass density of plate material, υ_3 is the out-of-plane displacement; a and b are the dimensions of the plate. Using (2) and (4) in expressions for potential and kinetic energy yields:

$$U = \frac{1}{2}\frac{4Db}{a^3}\sum_{r=1}^{p}\sum_{s=1}^{p} q_r q_s \int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{\partial^2 \Psi_r}{\partial \xi_1^2}\frac{\partial^2 \Psi_s}{\partial \xi_1^2}\right) + \frac{1}{(b/a)^4}\left(\frac{\partial^2 \Psi_r}{\partial \xi_2^2}\frac{\partial^2 \Psi_s}{\partial \xi_2^2}\right)\right.$$
$$\left. + \frac{\nu}{(b/a)^2}\left(\frac{\partial^2 \Psi_r}{\partial \xi_1^2}\frac{\partial^2 \Psi_s}{\partial \xi_2^2} + \frac{\partial^2 \Psi_s}{\partial \xi_1^2}\frac{\partial^2 \Psi_r}{\partial \xi_2^2}\right) + \frac{2(1-\nu)}{(b/a)^2}\left(\frac{\partial^2 \Psi_r}{\partial \xi_1 \partial \xi_2}\frac{\partial^2 \Psi_s}{\partial \xi_1 \partial \xi_2}\right)\right] d\xi_1 d\xi_2$$
$$= \frac{1}{2}\boldsymbol{q}_e^T \boldsymbol{K}_e \boldsymbol{q}_e \tag{8}$$

$$T = \frac{1}{2}\frac{\rho h a b}{4}\sum_{r=1}^{p}\sum_{s=1}^{p}\dot{q}_r \dot{q}_s \int_{-1}^{1}\int_{-1}^{1}\Psi_r \Psi_s d\xi_1 d\xi_2 = \frac{1}{2}\dot{\boldsymbol{q}}_e^T \boldsymbol{M}_e \dot{\boldsymbol{q}}_e \tag{9}$$

The (r, s) component of the stiffness matrix is defined as:

$$(K_e)_{r,s} = \frac{4Db}{a^3}\int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{\partial^2 \Psi_r}{\partial \xi_1^2}\frac{\partial^2 \Psi_s}{\partial \xi_1^2}\right) + \frac{1}{(b/a)^4}\left(\frac{\partial^2 \Psi_r}{\partial \xi_2^2}\frac{\partial^2 \Psi_s}{\partial \xi_2^2}\right)\right.$$
$$\left. + \frac{\nu}{(b/a)^2}\left(\frac{\partial^2 \Psi_r}{\partial \xi_1^2}\frac{\partial^2 \Psi_s}{\partial \xi_2^2} + \frac{\partial^2 \Psi_s}{\partial \xi_1^2}\frac{\partial^2 \Psi_r}{\partial \xi_2^2}\right) + \frac{2(1-\nu)}{(b/a)^2}\left(\frac{\partial^2 \Psi_r}{\partial \xi_1 \partial \xi_2}\frac{\partial^2 \Psi_s}{\partial \xi_1 \partial \xi_2}\right)\right] d\xi_1 d\xi_2 \tag{10}$$

The r, s entry of element's mass matrix is defined as:

$$(M_e)_{r,s} = \frac{\rho h a b}{4}\int_{-1}^{1}\int_{-1}^{1}\Psi_r \Psi_s d\xi_1 d\xi_2 \tag{11}$$

Due to the implemented natural coordinate system and trigonometric shape functions the integrals in (10) and (11) can be evaluated in a close form.

The formulation for modelling the in-plane motions of the plates must be consistent with the one for out-of-plane, but the hierarchical functions are not used. In order to fulfill the continuity requirements, the 12 DOF membrane element was developed. The shape functions are defined as a product of third order polynomials (see (5)) for one of co-ordinates and first order polynomials for the second co-ordinate. This formulation assure that the displacement field of two or more perpendicular elements will be continuous providing the hierarchical co-ordinates having non-zero displacement along the common edge are clamped.

3 Numerical Models

Two numerical methods are used in this study. The h-version of FEM is used as a benchmark (ABAQUS [13]). This results are compared with the application of hierarchical FEM developed in this paper. The comparison of the performance of h-version and hierarchical version of FEM will be presented on two examples, namely two perspex plates joined at right angle (previously presented in [14]) and a I-shaped plate girder made of five steel plates, see Figure 1.

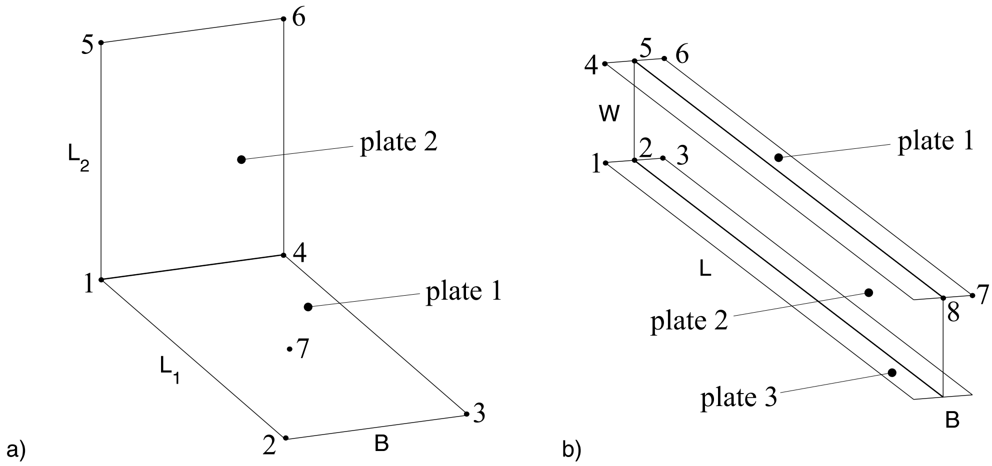

Fig. 1. Models used in analysis; a) two plates; b) plate girder

The numerical values used in analysis for the two plate corner: L_1=1m, L_2=0.5 m, B=0.39 m, thickness 0.01 m. Material data: Young's modulus 4.9 $\times 10^9$ N/m^2, Poisson ratio 0.25, density 1180 kg/m^3; all data after [14]. The plates are simply supported along edges, including the common edge, so that the energy transmission is possible trough rotational motions.

The accuracy of the results require that the displacement field of the structure is approximated accurately by the shape functions of the numerical model. In the h-version of FEM this is usually translated into the number of elements of certain order per wavelength (λ) of elastic deformation of the structure. For the first order interpolation functions (4-node plate element) the minimum number of elements is usually taken as 8. In the following it is required that the model has at least 8 first order elements per wavelength. Thin plate can support three types of elastic waves, namely quasi-longitudinal waves (denoted by subscript L), shear waves (denoted by subscript S) and bending waves (denoted by subscript S). The wavelengths for the three types of waves in plate under harmonic motions of frequency f are given by [15]:

$$\lambda_L = \frac{1}{f}\sqrt{\frac{E}{(1-\nu^2)\rho}} \; ; \; \lambda_S = \frac{1}{f}\sqrt{\frac{E}{2(1+\nu)\rho}} \; ; \; \lambda_B = \sqrt{\frac{2\pi}{f}}\sqrt[4]{\frac{Eh^2}{12(1-\nu^2)\rho}} \quad (12)$$

where E is the elasticity modulus, ν is the Poisson ratio, ρ is the mass density and h is the plate thickness.

The FEM models consist of rectangular four node thin plate elements (S4R5 in ABAQUS element library). Therefore, the criteria of 8 elements per wavelength will be used in comparison with the shortest of wavelengths. The hierarchical models consist of rectangular hierarchical elements with variable (third and above) order of internal co-ordinates for modelling of out-of-plane (bending) displacement and third order interpolation for in-plane (longitudinal and transverse) displacement. Therefore, the global mesh of elements have to fulfill the 4 element/wavelength criteria for the transverse waves λ_T (equivalent to 8 elements of first order), and the order of interpolation function for out-of-plane component can be adjusted for the wavelength of bending waves λ_B. Two frequency ranges, namely 10-600 Hz and 10-2500 Hz are considered. The wavelengths λ_B and λ_T are calculated at maximum frequencies in each range and included in Table 1. Based on this information the appropriate models are created and the details given in the table. For the sake of comparison the order of interpolation - N_i, and the total number of degrees of freedom in the model - DOF are also included.

The numerical values used in the analysis for the plate girder: L=1m, W=0.1 m, B=0.06 m, thickness 0.005 m; material data: Young's modulus 2.1 $\times 10^1 1$ N/m^2, Poisson ratio 0.3, density 7800 kg/m^3. The girder is clamped along edges 1-2, 2-3, 2-5, 4-5 and 5-6. In this case there are additional wave types that should be considered. Apart from the bending and transverse waves of plates the whole structure can behave as a beam. The bending of the girder in a beam-like mode would involve in-plane deformation and thus the global element mesh should be created based on the shorter of two wavelengths. In similar way as for two plate corner, the frequency range, wavelengths and details of the models are summarised in Table 2.

Table 1
Models of Plate Corner.

Freq [Hz]	λ_B [m]	λ_T [m]	No of El. Along L_1	L_2	B	Order of Int N_i	Model Tag	DOF
600	0.25	2.15	24	12	10	1	c-FEM-1	2442
600	0.25	2.15	4	2	2	3	c-HFEM-1-3	77
600	0.25	2.15	4	2	2	4	c-HFEM-1-4	135
600	0.25	2.15	4	2	2	5	c-HFEM-1-5	217
2500	0.12	0.52	48	24	20	1	c-FEM-2	9198
2500	0.12	0.52	8	4	4	3	c-HFEM-2-3	295
2500	0.12	0.52	8	4	4	4	c-HFEM-2-4	531
2500	0.12	0.52	8	4	4	5	c-HFEM-2-5	863

Table 2
Models of Plate Girder

Freq [Hz]	λ_B^p [m]	λ_T^p [m]	λ_B^b [m]	No of El. Along L	W	B/2	Order of Int N_i	Model Tag	DOF
600	0.28	5.36	0.47	24	3	1	1	g-FEM-1	1200
600	0.28	5.36	0.47	16	3	1	3	g-HFEM-1	772
2500	0.14	1.28	0.23	48	6	2	1	g-FEM-2	4410
2500	0.14	1.28	0.23	30	3	1	3	g-HFEM-2-3	2500
2500	0.14	1.28	0.23	30	3	1	4	g-HFEM-2-4	

4 Energy Flow

Since both methods are deterministic the of energy flow within a structure due to the harmonic point excitation will be sufficient to compare both methods. If the energy levels due to the excitation in one position do agree, than the spatial average over excitation points would agree, too. The commercial package ABAQUS was used in the finite element analysis. The output in terms of velocity amplitude at node n in direction k V_n^k and phase angle ϕ_n^k relative to excitation due to the point harmonic force can be calculated at each frequency (number of distinct frequencies within a band). The spatial average of the squared velocity amplitude $< V^2 >= 1/N \sum_{n=1}^{N} V_n^2$ can be calculated for each plate containing N nodes separately. The total energy of a plate equals the maximum value of kinetic energy or:

$$E_{FEM} = \frac{1}{2} m < V^2 > \tag{13}$$

where m is the mass of the plate. In the hierarchical formulation presented in this paper the total energy (equal maximum kinetic energy) is calculated directly from (9) using the amplitude of the (complex) co-ordinate q, i.e.

$$E_{HFEM} = \frac{1}{2}\omega^2 |\boldsymbol{q}|^T \boldsymbol{M} |\boldsymbol{q}|. \tag{14}$$

This difference is substantial for the hierarchical formulation since the element size might be great and calculating spatial average would otherwise require additional interpolation of displacement field at points within element.

The damping is accounted for by introducing the complex elasticity modulus $E_d = E(1 + i\eta)$ in both methods of analysis, with the damping loss factor η=0.001. This low value of damping might be easily encountered in the steel structures and it is one of the reasons why the statistical methods that predict only the mean response of the structure might not be used in analysis.

4.1 Two-plate-corner

The first comparison regards the space averaged energies of plates 1 and 2 forming the corner, see Figure 1 at frequencies below 600 Hz when excited by the point harmonic force acting at point 7 (at the centre of plate 1) normal to the surface. Models c-FEM-1 and c-HFEM-1-N_i are used in this calculations, see Table 1, where N_i signifies the order of interpolation functions for out-of-plane motions. The results for the energy plate one (E_1), plate two (E_2) and energy ratio (E_2/E_1) are compared in Figure 2. There is generally a good agreement between FEM and HFEM results below 200 Hz. Above 200 Hz the hierarchical model with third order interpolation functions is not reliable. Increasing the order to fourth and fifth one improves the convergence of natural frequencies considerably, when comparing with the FEM model. Both hierarchical models have below ten times less DOF than FEM model and result in good energy estimations at frequency of interest (in fact better than FEM considering the convergence of natural frequencies).

In the second example the frequency range is increased to 2500 Hz. The models used are c-FEM-2 and c-HFEM-2, see Table 1. The comparison of energy ratios between energy in driven plate and energy in receiving plate is presented in Figure 3. The improvement in the predictions follows the increase of the order of interpolation functions (compare results for c-HFEM-2-3, c-HFEM-2-4 and c-HFEM-2-5). Again the use of hierarchical FEM reduces the number of DOF by the factor of 10 (see, Table 1).

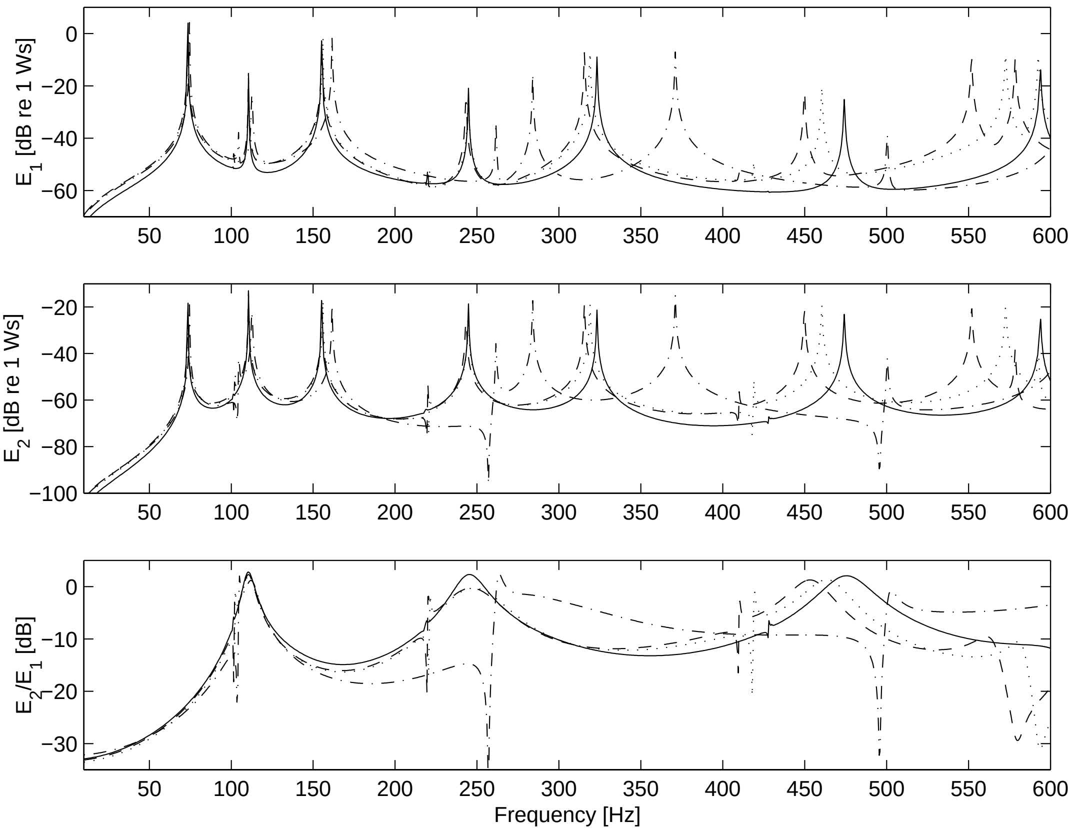

Fig. 2. Energy distribution (out-of-plane) in two-plate-corner, ——, c-FEM-1; - · · · ·, c-HFEM-1-3; · · ·, c-HFEM-1-4; - - -, c-HFEM-1-5

4.2 Five-plate-girder

The calculations for the five-plate-girder are performed in similar manner that the ones for two-plate-corner. The girder is clamped and excited at the free end, at point 7, see Figure 1 in the direction normal to the plate 1. First the frequency range below 700 Hz is considered. The energies out-of-plane motions of plates 1 (excited) and 2 are presented in Figure 4. The response in the frequency range considered is dominated by the beam-like modes (coupled in-plane motions and out-of-plane motions of perpendicular plates) of the girder, and therefore no substantial improvement can be achieved by increasing the interpolation functions. However, despite errors in the natural frequencies, the both energies, i.e. E_1 and E_2 and the energy ration agree between two results.

In the second calculation the frequency range is increased to 2500 Hz. The energies of plates 2 and 3 are presented in Figure 5.

At this range the bending motions of the plates contribute considerably to the

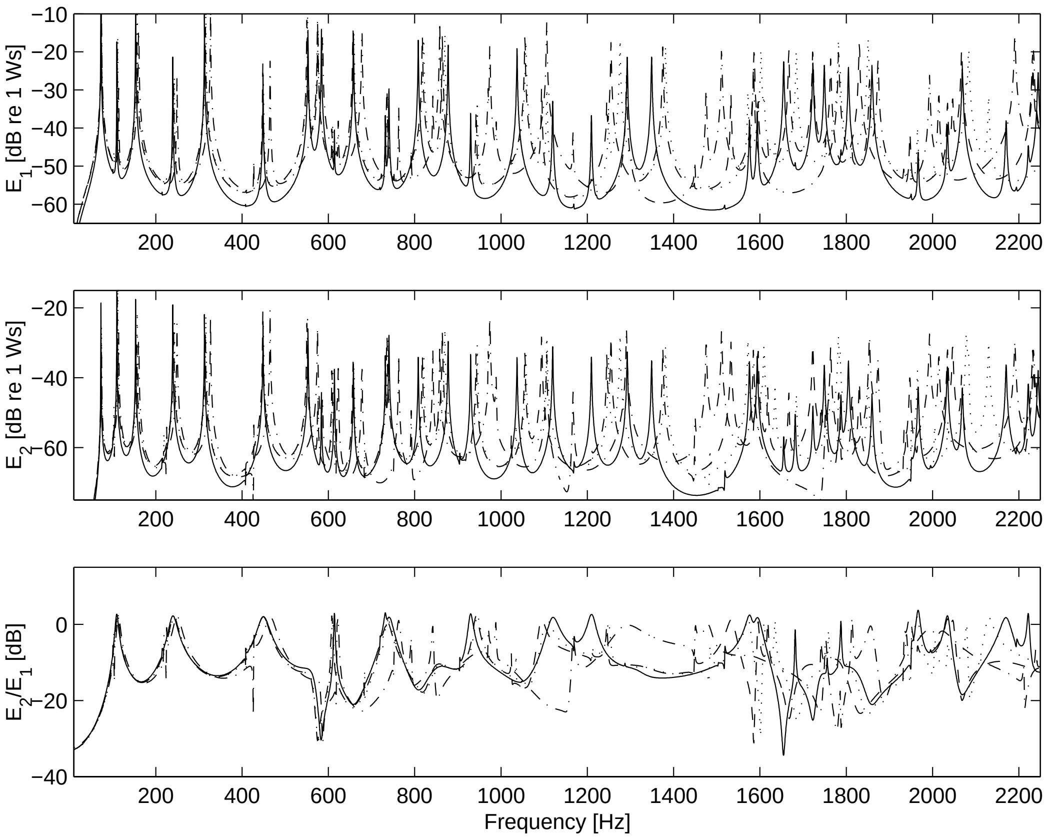

Fig. 3. Energy distribution (out-of-plane) in two-plate-corner, —–, c-FEM-2; - · - ·, c-HFEM-2-3; · · · , c-HFEM-2-4; - - -, c-HFEM-2-5

energy and the improvement achieved by adding hierarchical co-ordinates is clear.

5 Conclusions

This paper presents the application of hierarchical formulation of finite element method to modelling energy distribution within plate assemblies due to harmonic loading. The results are compared with standard FEM. The presented formulation increase the efficiency of FEM in terms of model size. The rate of improvement depends highly on the ratio of wavelengths of different wave types excited in the structure.

If the wavelengths of out-of-plane waves are much shorter than that of in-plane waves, like in the case of two-plate-corner, the increase in efficiency measured in terms of DOF is substantial. However, if the wavelengths are of comparable

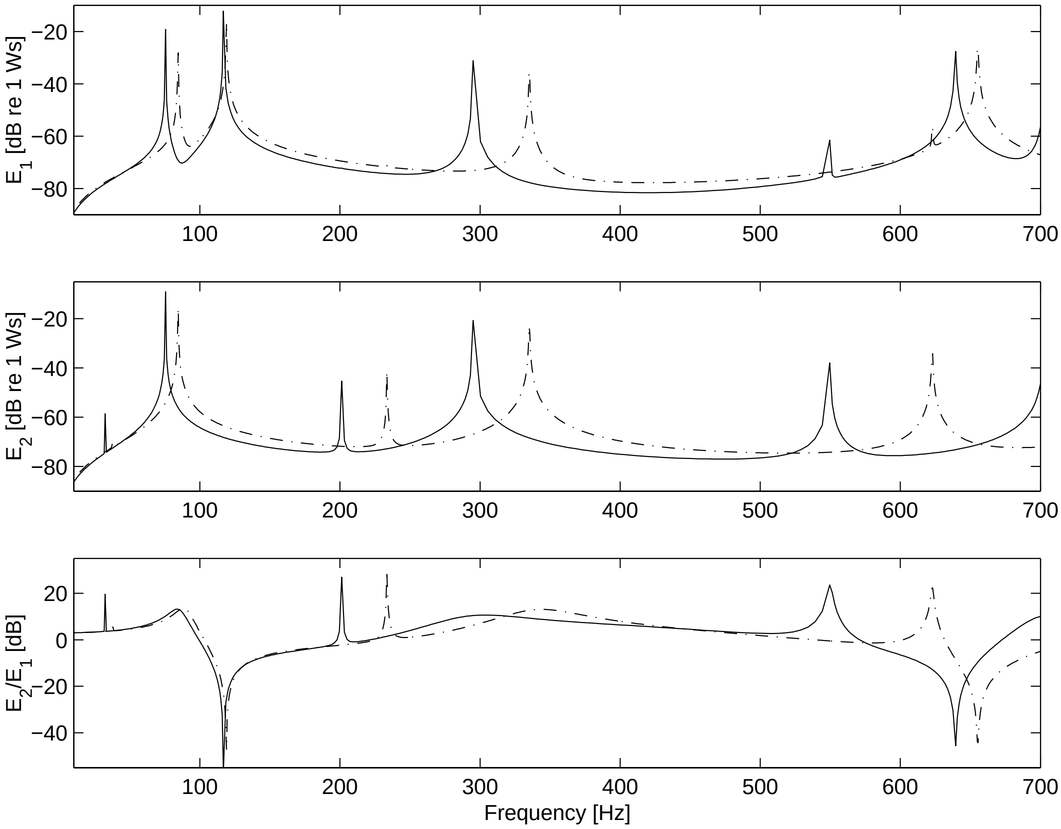

Fig. 4. Energy distribution (out-of-plane) in five-plate-girder, —–, g-FEM-1; - · - ·, g-HFEM-1-3.

lengths or the structure undergoes motions that couple in-plane and out-of-plane deformation one can not expect the hierarchical formulation to give better results. This was a case of the five-plate-girder. It is because the main advantage of the proposed formulation lies in efficient use of co-ordinates.

It can also be seen from the presented results, that the use of hierarchical formulation have an additional property, that the use of higher order co.ordinates is equivalent with the use of higher order modes, i.e. only the response above certain frequency is affected by the use of higher order hierarchical functions.

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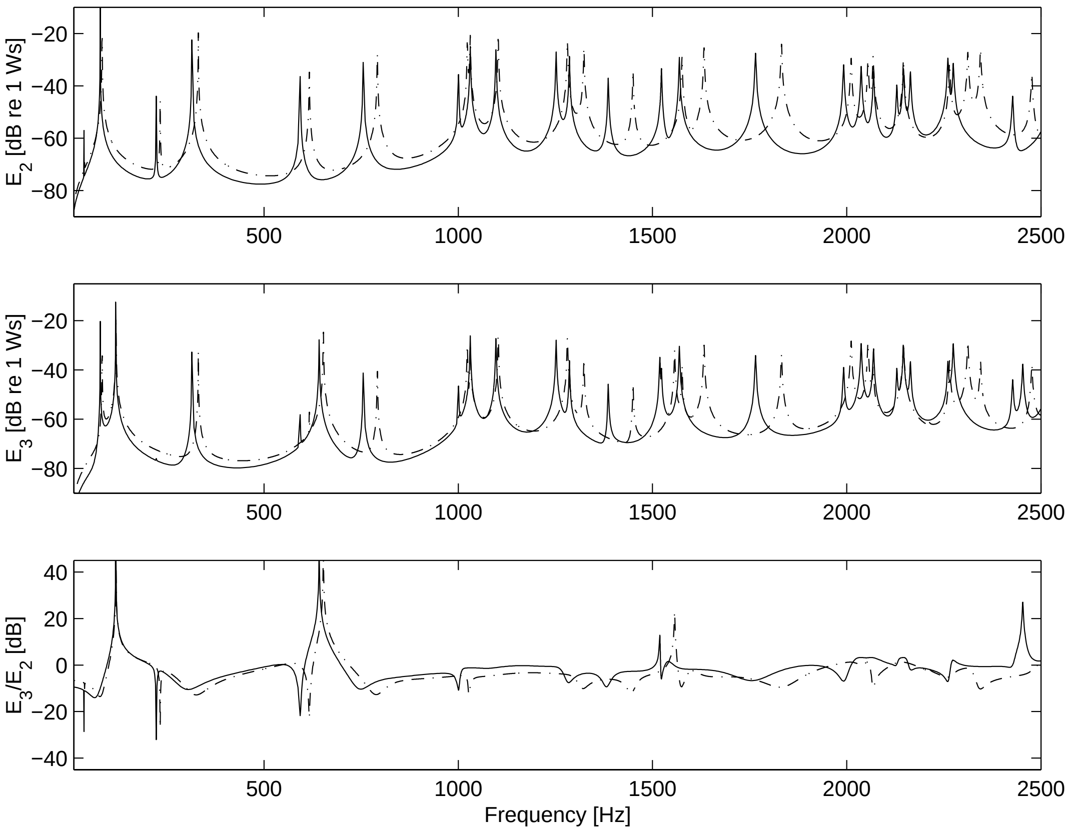

Fig. 5. Energy distribution (out-of-plane) in five-plate-girder, —–, g-FEM-2; - · - ·, g-HFEM-2-3; · · ·, g-HFEM-2-4;.

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