Boundary Element Analysis of the Stationary Response from a Moving Force on an Elastic Half-Space

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Abstract
The paper deals with the boundary element method formulation of the steady-state vibration wave propagation through elastic media due to a source moving with constant velocity. The Green’s function for the three-dimensional full-space is formulated in a local frame of reference following the source. This is appropriate for the analysis of, for example, ground borne noise from railway or road traffic. The frequency-domain fundamental solution is derived from the corresponding time-domain solution by means of Fourier transformation. To obtain a closed form solution, a part of the time-domain kernel functions are approximated, but the error which is introduced in this way, is insignificant. Numerical examples are given for a moving rectangular load on an elastic half-space. The results from a boundary element code based on the derived Green’s function are compared with a semi-analytic solution.

Keywords: Moving source; Convection; Elastodynamics; Green’s function; Frequency domain.

1. Introduction
Over the last decades, the analysis of vibration wave propagation from traffic has been of increasing interest, in particular in densely populated areas. Divers problems, such as free field vibrations from road traffic and ground borne noise from metro tunnels, have been analysed using a variety of techniques. For the analysis of vehicles and loads moving at low velocities, the uniform convection may be disregarded in the model without significant loss of accuracy. However, for high velocities, the convection must be included in the mathematical model. This is particularly the case within the field of railway engineering, because modern passenger trains may operate at speeds, which approach, or even exceed, the propagation velocity of waves in the soil. High-speed trains was one of the main issues at the International Workshop Wave 2000 [1].

A railway track may be modelled as a beam resting on an elastic foundation. This rather crude approach was taken by Andersen et al. [2], who investigated the response of a track and a moving vehicle due to surface roughness of the track. However, to get more realistic results than those obtained with a beam on a Kelvin foundation, a model of the soil is needed, which takes into account the propagation of volume and surface waves. For simple geometries of the ground, an analytical solution can be derived. Singh and Kuo [3] studied the response from a moving circular load on the surface of a half-space in fixed coordinates. Jones et al. [4] gave a formulation for a moving rectangular harmonic source on the surface of a homogeneous half-space in the frequency-wavenumber domain and in a moving frame of reference using the Galilean coordinate transformation. The model was extended by Sheng et al. [5] to the analysis of a railway track on top of a layered ground with horizontal surface and interfaces. An analytical solution for the response of a viscoelastic layer and a half-space due to a moving load was given by Dieterman and Metrikine [6, 7], whereas a two-dimensional analysis of a moving train in a tunnel was carried out by Metrikine and Vrouwenvelder [8]. In both cases, the analysis was carried out in a fixed frame of reference.

For complicated geometries, a numerical solution is needed. The finite element method (FEM) is widely used for wave propagation problems, even though absorbing or transmitting boundary conditions are needed at the artificial boundaries. Krenk et al. [9] formulated local boundary conditions for the FEM analysis of the elastodynamic full-space problem in the moving frame of reference, while boundary conditions for the beam-on-elastic-foundation problem where formulated by Andersen et al. [10]. However, for the analysis of a variety of problems with ground-borne vibration, the boundary element method (BEM) is superior to the FEM due to its inherent capability of radiating waves. Unfortunately, before the BEM can be applied to a given prob-
lem, the Green’s function, or fundamental solution, must be known. For the analysis of three-dimensional problems, the full-space or the half-space solution have both been used. Both the time- and the frequency domain BEM solutions to the elastodynamic problem are well described in the literature for the analysis in a fixed frame of reference. An extensive review of the application of the BEM to various problems in elastodynamics was given by Beskos [11].

Lombaert et al. [12] studied the vertical response from road traffic, using a boundary element formulation with so-called relaxed boundary conditions for the soil, and performing the analysis in the frequency an wave number domains. A time-domain boundary element analysis of a load moving with various uniform velocities on top of a half-space was carried out by Chouw and Pflanz [13], using the full-space fundamental solution in the fixed frame of reference. Alternatively, Tadau and Kausel [14] formulated a so-called two-and-a-half-dimensional, or quasi-two-dimensional, fundamental solution, which may be used for the analysis of structures with a two-dimensional geometry, e.g. a railway tunnel/track or a strip foundation. Here, a transformation into Laplace or Fourier domain is carried out for one of the coordinate directions. This method was also addressed by Hirose [15], who further discussed different Green’s functions for the three-dimensional BEM analysis of a homogeneous or horizontally layered half-space subject to a moving load.

For a computation of the stationary response to a moving load, a fixed-frame-of-reference description is inadequate. Here a large model is required to allow the stationary condition to fully evolve, when the load is moving relatively to the mesh. In several problems, the geometry and the material behaviour are close to being constant along the direction, in which the load is moving. For this class of problems, the quasi-two-dimensional model may be used. Alternatively, the problem may be described in the Cartesian space, but in a moving frame of reference following the load. This way a significantly smaller model can be used than is necessary in the fixed frame of reference. In a moving-frame-of-reference model, convection is introduced in the solid medium in the same way as in the Eulerian description of a fluid in the general Navier-Stokes equation. An advantage of the moving-frame-of-reference description is that the element mesh follows the load or vehicle. Hence, any mesh refinement, which would be necessary around the vehicle, can be made once and for all. The original three-dimensional formulation of the problems is kept in this method. Hence, a model with more degrees of freedom arise than is obtained by quasi-two-dimensional analysis. However, no Fourier or Laplace transformation is necessary over the spatial coordinates.

To the convection-dominated acoustic problem, a closed-form solution can be found in the frequency domain. Wu and Lee [16] derived the fundamental solution for subsonic flow and for a harmonic source by application of the frequency-domain equivalent of the Lorenz transformation and subsequent transformation back into Galilean coordinates. However, for the elastodynamic problem in the moving frame of reference, a closed-form solution cannot be obtained in the frequency domain, because the Fourier transform of the convection-dominated Navier equation cannot be broken down to a number of Helmholtz equations with spherical symmetry of both the inhomogeneity and the differential operators.

In the present paper, a three-dimensional boundary element formulation is given for the stationary response of an elastic medium in a local frame of reference following a moving harmonic load. Firstly, the time-domain fundamental full-space solutions for the displacement and surface traction in the convected coordinate system are discussed. The time-domain solutions were formulated by Rasmussen [17]. Next, the moving-frame-of-reference equivalent to the Somigliana Identity is derived. Subsequently, the frequency-domain fundamental solutions are obtained by means of Fourier transformation of the time-domain Green’s functions. To the best of the authors’ belief, this approach has not before been taken. So far, the use of the moving-frame-of-reference description in the frequency domain has been limited to analysis in the wave number domain.

The theory described in the present paper has been implemented as an extension to the software program BEASTS, originally developed for analysis in the fixed frame of reference, see [18]. Here, boundary elements with biquadratic interpolation are used and a coupling between multiple BE domains is carried out in a macro-finite-element sense. A numerical example is given, in which a homogeneous and a layered half-space are subject to a harmonic rectangular load, moving along the surface at various subsonic velocities. The quality of the solution is estimated by a comparison with the semi-analytic solution given by Jones et al. [4].

2. Theory

Consider an isotropic, homogeneous, linearelastic body, \( \Omega \), bounded by the surface \( \Gamma \). In a fixed Cartesian space, let \( u_j(x, t) \) and \( b_j(x, t) \) denote the displacement field and the load per unit mass, respectively, where \( x \) is the coordinate vector with the components \( x_j, j = 1, 2, 3, \) and \( t \) is time. In terms of the Lamé constants \( \lambda \) and \( \mu \) and the mass density \( \rho \), the equation of motion in the time domain, i.e. the Navier equation, may be written,

\[
(\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j} u_j(x, t) + \mu \frac{\partial^2}{\partial x_j \partial x_i} u_i(x, t) + \rho b_j(x, t) = \rho \frac{\partial^2}{\partial t^2} u_j(x, t),
\]

(1)

where the following boundary and initial conditions apply,

\[
\begin{align*}
  u_j(x, t) &= \bar{u}_j(x, t) \quad \text{for} \quad x \in \Gamma_1, \\
  p_j(x, t) &= \bar{p}_j(x, t) \quad \text{for} \quad x \in \Gamma_2,
\end{align*}
\]

(2)

\( \Gamma = \Gamma_1 \cup \Gamma_2 \).
The coordinate transformation is illustrated in Figure 1.

\[
\begin{align*}
&\begin{align*}
u_j(x,0) &= u_j^0(x) \quad \text{for } x \in \Omega \\
\frac{\partial}{\partial t} u_j(x,0) &= v_j^0(x) \quad \text{for } x \in \Omega \end{align*},
\end{align*}
\]

(3)

Here \( \sigma(x,t) \) is the traction, which is defined on the surface \( \Gamma \) in terms of the Cauchy stress tensor and the outward unit normal \( \mathbf{n} \) having the components \( \sigma_{ij}(x,t) \) and \( n_j(x) \), respectively. In Eq. (1) and subsequent equations, the summation convention applies, i.e. summation is performed over repeated indices.

For the purpose of analysing moving loads, a transformation is carried out from the fixed Cartesian \((x_1,x_2,x_3)\)-coordinate system into the co-directional \((\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)\)-coordinate system following the load. The load is assumed to be moving at the constant velocity \( \mathbf{v} \) with components \( v_j \) and magnitude \( |\mathbf{v}| = v \). Thus the Galilean transformation,

\[
\tilde{x} = x - \mathbf{v}(t - \tau),
\]

(4)

applies, where the two coordinate systems coincide at \( t = \tau \). The coordinate transformation is illustrated in Figure 1. Partial derivatives in the two coordinate systems are related in the following manner,

\[
\frac{\partial}{\partial x_j} = \frac{\partial}{\partial \tilde{x}_j} \quad \text{with} \quad \left| \frac{\partial}{\partial \tilde{x}_j} \right|_{x_j} = \frac{\partial}{\partial t} \bigg|_{x_j} - v_j \frac{\partial}{\partial \tilde{x}_j}.
\]

(5)

\[
\frac{\partial^2}{\partial t^2} \bigg|_{x_j} = \frac{\partial^2}{\partial \tilde{x}_j^2} + v_j v_{k \tilde{x}_j} \frac{\partial^2}{\partial \tilde{x}_j \partial \tilde{x}_k}.
\]

(6)

Hence, the equation of motion in the moving frame of reference becomes

\[
(\lambda + \mu) \frac{\partial^2}{\partial \tilde{x}_j \partial \tilde{x}_j} u_{ij}(\tilde{x},t) + \mu \frac{\partial^2}{\partial \tilde{x}_j \partial \tilde{x}_k} u_{jk}(\tilde{x},t) + \rho \frac{\partial b_i(\tilde{x},t)}{\partial \tilde{x}_j} = \rho \frac{\partial^2}{\partial \tilde{x}_j \partial \tilde{x}_j} u_i(\tilde{x},t).
\]

(7)

The time-domain Green’s function for the displacement, \( u_{ij}(x,t; y, \tau) \), is the solution to Eq. (1) for a unit impulse point force acting in the \( x_j \) direction at the time \( t = \tau \) and at the point \( y \), that is a load in the form \( \rho b_i(\tilde{x},t; y, \tau) = \delta(x - y) \delta(t - \tau) \delta_\tilde{x} \) applied to an infinite continuum, which is a rest at time \( t = \tau \). Here \( \delta(x) \) is the Dirac delta, or unit impulse, function and \( \delta_\tilde{x} \) is the Kronecker delta. Correspondingly, for a load moving with the constant velocity \( \mathbf{v} \), the displacement Green’s function \( \tilde{u}_i(\tilde{x},t; y, \tau) \) is the solution to Eq. (7) when the inhomogeneity is applied as a point force in the form

\[
\rho \tilde{b}_i(\tilde{x},t; \tilde{y}, \tau) = \delta(\tilde{x} - \tilde{y}) \delta(t - \tau) \delta_\tilde{x},
\]

(8)

or, in the fixed frame of reference,

\[
\rho \tilde{b}_i(\tilde{x}; \tilde{y}, \tau) = \delta(x - y)(t - \tau) \delta_\tilde{x}.
\]

(9)

As \( \delta(t - \tau) = 0 \) when \( t \neq \tau \), the velocity term vanishes, whereby it becomes evident that

\[
\rho \tilde{b}_i(\tilde{x}; \tilde{y}, \tau) = \rho \tilde{b}_i(\tilde{x}; \tilde{y}, \tau).
\]

(10)

The solution to Eq. (1) for a point excitation of this kind is \( u_i^0(x,t; y, \tau) \). Thus the Green’s function for the moving load may be expressed in terms of the Green’s function for the stationary load as

\[
u_i^0(\tilde{x}, t; \tilde{y}, \tau) = u_i^0(\tilde{x} + \mathbf{v}(t - \tau), t; \tilde{y}, \tau).
\]

(11)

Likewise, in the moving frame of reference, the fundamental solution for the surface traction in case of a moving load may be formulated as

\[
\tilde{p}_i^0(\tilde{x}, t; \tilde{y}, \tau) = p_i^0(\tilde{x} + \mathbf{v}(t - \tau), t; \tilde{y}, \tau),
\]

(12)

where \( p_i^0(x,t; y, \tau) \) is the Green’s function for a stationary point force (in the fixed frame of reference). These relationships were derived by Rasmussen [17].

The time-domain Green’s functions for the displacement and surface traction due to a stationary source in a fixed frame of reference are well described in the literature and may be found, for example, in [19]. Making use of Eqs. (11) and (12), the Green’s functions for the moving load can be established. The nature of the solution depends on the velocity of convection relative to the phase velocities of the wave propagation in the material. In an isotropic, homogeneous, linear elastic continuum, two kinds of volume waves may exist, namely pressure (P-) and shear (S-) waves. The phase velocities are

\[
c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_S = \sqrt{\mu/\rho}.
\]

(13)

In the case of subsonic convection with respect to both the P- and S-waves, i.e. \( v < c_S < c_P \), one wavefront of either type will occur at any observation point \( \tilde{x} \) due to excitation at the source point \( \tilde{y} \), see Fig. 2a. This corresponds to the situation for a stationary load in the fixed frame of reference. Hence, the fundamental solutions for subsonic convection
become,
\[
\begin{align*}
\tilde{u}_d^i(\tilde{x}, t; \tilde{y}, \tau) &= -\frac{1}{4\pi \rho} \left( \frac{\delta g}{r^3} - \frac{3r_j n_j}{r^3} \right) (t - \tau) \\
&\times \left[ \frac{1}{r_c P} \left( r_j n_j + 2n_j n_i + \delta g r_j n_j \right) - \frac{r_j n_j}{r^5} \right] \delta \left( t - \tau - \frac{r}{r_c P} \right) \\
&+ \frac{1}{4\pi r_c P} \left( \frac{\delta g}{r^3} - \frac{3r_j n_j}{r^3} \right) \delta \left( t - \tau - \frac{r}{r_c P} \right) \\
&+ \frac{1}{4\pi r_c S} \left( \frac{\delta g}{r^3} - \frac{3r_j n_j}{r^3} \right) \delta \left( t - \tau - \frac{r}{r_c S} \right),
\end{align*}
\]
(14)
\[
\tilde{p}_d^i(\tilde{x}, t; \tilde{y}, \tau) = \frac{1}{4\pi} \left[ 2 \frac{c_S^2}{c_p^2} \left( \frac{r_j n_j + 2n_j n_i + \delta g r_j n_j}{r^5} \right) - \frac{r_j n_j}{r^3} \right] \delta \left( t - \tau - \frac{r}{r_c P} \right) \\
+ \frac{c_S^2}{4\pi} \left( 6 \frac{r_j n_j + 2n_j n_i + \delta g r_j n_j}{r^5} - 30 \frac{r_j n_j}{r^3} \right) \delta \left( t - \tau - \frac{r}{r_c P} \right) \\
&- \frac{1}{4\pi c_p} \left[ 2 \frac{c_S^2}{c_p^2} \left( \frac{r_j n_j - r_j n_i}{r^4} + \frac{n_j n_i}{r^2} \right) + \frac{r_j n_j}{r^2} \right] \delta \left( t - \tau - \frac{r}{r_c P} \right) \\
&- \frac{1}{4\pi c_S} \left( 3r_j n_i + 2n_j n_i + 3\delta g r_j n_j \right) \delta \left( t - \tau - \frac{r}{r_c S} \right) \\
&- 12 \frac{r_j n_j}{r^5} \delta \left( t - \tau - \frac{r}{r_c S} \right),
\]
(15)
where \( r_j = r_j(\tilde{x}, t; \tilde{y}, \tau) \) are the components of the distance vector \( \mathbf{r}(\tilde{x}, t; \tilde{y}, \tau) = \tilde{x} + \mathbf{v}(t - \tau) - \tilde{y} \) between the source point \( \tilde{y} \) at time \( \tau \) and the observation point \( \tilde{x} \) at time \( t \), and
\[
r = \sqrt{\tilde{x}^2 + \tilde{y}^2} = |\mathbf{r}(\tilde{x}, t; \tilde{y}, \tau)|.
\]
(16)
Here \( \tilde{r}_j = \tilde{x}_j - \tilde{y}_j, \tilde{r} = |\tilde{\mathbf{r}}| \). Note that \( n_j = n_j(\tilde{y}) \) are the components of the unit outward normal, \( \mathbf{n}(\tilde{y}) \), to the surface \( \Gamma \) at the source point, not at the observation point \( \tilde{x} \). Furthermore, \( H(\cdot) \) is the Heaviside unit step function defined as,
\[
H(x) = \begin{cases} 
1 & \text{for } x \geq 0 \\
0 & \text{for } x < 0 
\end{cases},
\]
(17)
and \( \delta(\cdot) \) is the time derivative of the Dirac delta function. The following definitions apply for any function \( f(t) \), which is continuous respectively differentiable, at time \( t \),
\[
\begin{align*}
\int_{-\infty}^{\infty} \delta(t - \tau) f(\tau) \, d\tau &= f(t), \\
\int_{-\infty}^{\infty} \delta(t - \tau) f(\tau) \, d\tau &= \frac{\partial}{\partial t} f(t) \bigg|_{t=\tau}.
\end{align*}
\]
(18)
(19)
In Eqs. (14) and (15), the arguments of the Heaviside unit step functions are used to indicate that nothing happens before the arrival of the P-wave or after the arrival of the S-wave. Likewise, the arguments of the Dirac delta functions ensure that only at the passage of a P- or S-wavefront are there any contributions to the response. In particular \( t_P = r/c_p \) and \( t_S = r/c_P \) correspond to the arrival times for P- and S-waves, respectively, when a point load is applied at the position \( \tilde{y} \) and at the time \( \tau = 0 \). However, as \( r \) is time-dependent, the determination of \( t_P \) and \( t_S \) is not straightforward. In order to find the arrival times, a variable substitution in the form \( s = S_A(t) = t - r/c_A \) is introduced, where subscript \( A \) is either \( P \) or \( S \) so that \( c_A \) is equal to either \( c_P \) or \( c_S \). The inverse transformation \( t = T_A(s) \) is
obtained by a solution of the equation
\[ c_A^2 (T_A(s) - s)^2 = \{ r(x, T_A(s) : y, 0) \}^2 = r^2 + 2r v_j T_A(s) + v^2 \{ T_A(s) \}^2 \] (20)

The arrival times for P- and S-waves are identified as \( t_P = T_P(0) \) and \( t_S = T_S(0) \), respectively. Physically, only positive values of the arrival times are valid. For any subsonic convection velocity and for any location of the source and observation point, only one of the roots to Eq. (20) is positive when \( s = 0 \), namely
\[ T_A(s) = \frac{c_A^2 s + v_j r_j + \chi_A(s)}{c_A^2 - v^2}, \] (21)
where
\[ \chi_A(s) = \sqrt{v_j v_k r_j r_k + (c_A^2 - v^2) r^2 + v_A^2 v_j s (2r_j + v_j s)}. \]

\( T_A(s) \), as defined in Eq. (21), is a monotone increasing function of \( s \). Hence, a unique mapping exists between \( t \) and \( s \) in the case of subsonic convection.

When the convection is not subsonic with respect to both P- and S-waves, the problem becomes slightly more complicated. In case of transsonic convection, i.e. for \( c_S < v < c_P \), where \( v \) is the convection velocity and \( c_S \) and \( c_P \) are the speeds of sound at the source and the observation point, respectively. Points of the different types are illustrated in Fig. 2b as \( A, B \) and \( C \), respectively. The modifications to the Green’s functions, which are necessary in each individual case, are listed below.

- At a point \( \hat{x} \) lying on the surface of the Mach cone, only one positive root \( t_S \) exists. The corresponding S-wave arrives after an infinite duration of time. Hence, in practice, all terms with \( r/c_S \) appearing in the argument vanish. It is noted that in a numerical integration scheme in terms of Gauss-Legendre quadrature, which is usually adopted in the BEM, integration points are unlikely to lie exactly on the surface of the Mach cone.
- Outside the Mach cone, both roots are negative. Here no S-waves will arrive at the observation point. Therefore, all terms with \( r/c_S \) appearing in the argument vanish.

A similar procedure is necessary in the case of supersonic convection velocities, though here the P-wave terms must also be modified.

2.1 Boundary Integral Equation for the Convection-dominated Elastodynamic Problem

The boundary element and boundary integral formulation of the elastodynamic problem in a fixed frame of reference is based on the so-called Somigliana Identity, which defines a relationship between the physical state variables and the fundamental solutions on the surface \( \Gamma \) of the domain, see for example [19]. When a moving frame of reference is considered, the governing integral identity is changed due to the presence of convection. In this subsection, a moving-frame-of-reference equivalent to the Somigliana Identity is derived.

On the domain \( \Omega \) with boundary \( \Gamma \), let two states be given, which satisfy the convection dominated Cauchy equation at the times \( t = t_1 \) and \( t = t_2 \), respectively. For any point \( \hat{y} \) the following identities apply,
\[ \frac{\partial}{\partial y_j} \sigma_{ij}^{(1)}(\hat{y}, t_1) + \rho h_{ij}^{(1)}(\hat{y}, t_1) \]
\[ = \rho \left( \frac{\partial^2}{\partial t^2} - 2v_j \frac{\partial}{\partial y_j \partial t} + v_j v_k \frac{\partial^2}{\partial y_j \partial y_k} \right) u_i^{(1)}(\hat{y}, t_1), \] (22)
\[ \frac{\partial}{\partial y_j} \sigma_{ij}^{(2)}(\hat{y}, t_2) + \rho h_{ij}^{(2)}(\hat{y}, t_2) \]
\[ = \rho \left( \frac{\partial^2}{\partial t^2} - 2v_j \frac{\partial}{\partial y_j \partial t} + v_j v_k \frac{\partial^2}{\partial y_j \partial y_k} \right) u_i^{(2)}(\hat{y}, t_2). \] (23)

Multiplication of Eq. (22) with \( u_i^{(2)}(\hat{y}, t_2) \), multiplication of Eq. (23) with \( u_i^{(1)}(\hat{y}, t_1) \) and integration over the volume lead to the equations
\[ \int_{\Omega} \frac{\partial}{\partial y_j} \sigma_{ij}^{(1)}(\hat{y}, t_1) u_i^{(2)}(\hat{y}, t_2) d\Omega(\hat{y}) \]
\[ + \int_{\Omega} \rho h_{ij}^{(1)}(\hat{y}, t_1) u_i^{(2)}(\hat{y}, t_2) d\Omega(\hat{y}) \]
\[ - \int_{\Omega} \rho \left( \frac{\partial^2}{\partial t^2} - 2v_j \frac{\partial}{\partial y_j \partial t} + v_j v_k \frac{\partial^2}{\partial y_j \partial y_k} \right) u_i^{(1)}(\hat{y}, t_1) \]
\[ \times u_i^{(2)}(\hat{y}, t_2) d\Omega(\hat{y}) = 0, \] (24)
With $E_{ijkl}(\bar{y})$ denoting the elasticity tensor, the Cauchy stress tensor is written as 

$$\sigma_{ij}(\bar{y}, t) = E_{ijkl}(\bar{y}) \frac{\partial u_k}{\partial y_j}(\bar{y}, t).$$

Application of the divergence theorem to the first term in Eqs. (24) and (25) then provides,

$$\int_{\Omega} \frac{\partial}{\partial t} u_i^{(2)}(\bar{y}, t_2) \, d\Omega(\bar{y})$$

$$= \int_{\Omega} \rho \frac{\partial^2}{\partial t^2} u_i^{(2)}(\bar{y}, t_2) \, d\Omega(\bar{y}) - \int_{\Omega} \rho \left( \frac{\partial}{\partial t} \frac{\partial^2}{\partial y_j \partial t} + \frac{\partial^2}{\partial y_j \partial y_k} \right) u_i^{(2)}(\bar{y}, t_2) \times u_i^{(1)}(\bar{y}, t_1) \, d\Omega(\bar{y}).$$

(25)

The response in the second state must fulfill the causality condition that the medium has a quiescent past, i.e.

$$u_i^{(1)}(\bar{y}, t; \bar{y}, \tau) = 0, \quad \frac{\partial}{\partial \tau} u_i^{(2)}(\bar{y}, t; \bar{y}, \tau) = 0 \quad \text{for} \quad t \leq \tau,$$

(30)

since no load is applied before the delta spike at $t = \tau$. From Eqs. (8), (14) and (15), it is seen that the fundamental solution only depends on the arguments $t$ and $\tau$ through the difference $t - \tau$. Hence, the fundamental solution is invariant to time translation, i.e.

$$u_i^{(2)}(\bar{x}, t; \bar{y}, \tau) = u_i^{(2)}(\bar{x}, 0; \bar{y}, t - \tau) = \bar{u}_i^{(2)}(\bar{x}, t - \bar{y} ; 0).$$

(31)

Assume that the excitation $b_i(\bar{y}, \tau)$ has been acting for an infinite duration of time, so that no influence of initial conditions is observed at time $t$. Integration of Eq. (28) from $\tau = -\infty$ to $\tau = t$ then provides the identity

$$\int_{\Omega} \int_{-\infty}^{t+} \bar{u}_i^{(2)}(\bar{x}, t - \tau; \bar{y}, 0) \rho \frac{\partial}{\partial \tau} u_i^{(1)}(\bar{y}, \tau) \, d\Omega(\bar{y})$$

$$= \int_{\Omega} \rho \frac{\partial^2}{\partial t^2} \bar{u}_i^{(2)}(\bar{x}, t - \tau; \bar{y}, 0) \, d\Omega(\bar{y}) - \int_{\Omega} \rho \left( \frac{\partial}{\partial t} \frac{\partial^2}{\partial y_j \partial t} + \frac{\partial^2}{\partial y_j \partial y_k} \right) \bar{u}_i^{(2)}(\bar{x}, t - \tau; \bar{y}, 0) \times u_i^{(1)}(\bar{y}, \tau) \, d\Omega(\bar{y}).$$

(32)
is merely used to describe a uniform motion of the elastic medium relative to the applied load. Thus, the divergence theorem may be used on the convection terms on the left-hand side of Eq. (32) (twice to the second term). This way, the following relationship is achieved,

\[
\int_{t_1}^{t_2} \int_{-\infty}^{+} 2 v_j \dot{u}_n^j \left( \mathbf{x}, t - \tau; \mathbf{y}, 0 \right) \frac{\partial^2}{\partial x_j \partial \tau} u_t \left( \mathbf{y}, \tau \right) \, d\tau \, d\Omega(\mathbf{x})
- v_j v_k \ddot{u}_n^j \left( \mathbf{x}, t - \tau; \mathbf{y}, 0 \right) \frac{\partial}{\partial x_j \partial \tau} u_t \left( \mathbf{y}, \tau \right) \, d\tau \, d\Omega(\mathbf{x})
\]

The first term of the volume integral on the right-hand side of Eq. (33) is further integrated by parts with respect to time,

\[
\left[ \int_{t_1}^{t_2} \int_{t_1}^{t_2} 2 v_j \frac{\partial}{\partial x_j} \ddot{u}_n^j \left( \mathbf{x}, t - \tau; \mathbf{y}, 0 \right) \frac{\partial}{\partial \tau} u_t \left( \mathbf{y}, \tau \right) \, d\tau \, d\Omega(\mathbf{x}) \right]
+ \left[ \int_{t_1}^{t_2} \int_{-\infty}^{+} v_j v_k \frac{\partial}{\partial x_j \partial \tau} \ddot{u}_n^j \left( \mathbf{x}, t - \tau; \mathbf{y}, 0 \right) u_t \left( \mathbf{y}, \tau \right) \, d\tau \, d\Omega(\mathbf{x}) \right]
\]

use has been made of the causality condition \( \ddot{u}_n^j \left( \mathbf{x}, t; \mathbf{y}, \tau \right) = 0 \) for \( t \leq \tau \) along with the fact that the response in the physical field must vanish for \( \tau \to -\infty \). By partial integration and further application of Eq. (30), it may be proven that the acceleration terms on either side of Eq. (32) cancel out. Hence, with the use of Eqs. (33) and (34), the integral identity given in Eq. (32) may alternatively be written,

\[
\int_{t_1}^{t_2} \int_{-\infty}^{+} \ddot{u}_n^j \left( \mathbf{x}, t - \tau; \mathbf{y}, 0 \right) p_t \left( \mathbf{y}, \tau \right) \, d\tau \, d\Omega(\mathbf{y})
+ \int_{t_1}^{t_2} \int_{-\infty}^{+} 2 v_j v_n \ddot{u}_n^j \left( \mathbf{x}, t - \tau; \mathbf{y}, 0 \right) \frac{\partial}{\partial \tau} u_t \left( \mathbf{y}, \tau \right) \, d\tau \, d\Omega(\mathbf{y})
\]

use has been made of the causality condition \( \ddot{u}_n^j \left( \mathbf{x}, t; \mathbf{y}, \tau \right) = 0 \) for \( t < 0 \) so that it makes no difference whether the lower integration limit is taken as \( 0^- \) or \(-\infty\).

Carrying out the procedure in Eq. (35) for the remaining terms in Eq. (35) and disregarding the term \( e^{i\omega\tau} \), which

Here, volume integration is only carried out over the load terms. In case the convection velocity is allowed to be time-dependent, i.e., \( v = v(\tau) \), it should be noted that volume integrals will also arise for the convection acceleration term \( \partial v / \partial \tau \). Likewise, volume integration must be carried out over the initial conditions, when these are not identically equal to zero. In the present case, however, the aim is to establish the integral identity in the frequency domain. Here a time variation of the convection velocity and non-zero initial conditions are not allowed.

Assume a harmonic variation with time of the field and surface quantities in the physical state,

\[
\begin{align*}
\bar{u}_t \left( \mathbf{y}, \tau \right) &= U_t \left( \mathbf{y}, \omega \right) e^{i\omega \tau} \\
\bar{p}_t \left( \mathbf{y}, \tau \right) &= P_t \left( \mathbf{y}, \omega \right) e^{i\omega \tau} \\
\bar{b} \left( \mathbf{y}, \tau \right) &= E_b \left( \mathbf{y}, \omega \right) e^{i\omega \tau}
\end{align*}
\]

Here \( i = \sqrt{-1} \) is the imaginary unit and \( \omega = 2\pi f \) is a circular frequency corresponding to the physical frequency \( f \). Inserting the harmonically varying physical traction into the first integral term of Eq. (35) provides,

\[
\int_{-\infty}^{+} \int_{-\infty}^{+} \ddot{u}_n^j \left( \mathbf{x}, t - \tau; \mathbf{y}, 0 \right) P_t \left( \mathbf{y}, \tau \right) \, d\tau \, d\Omega(\mathbf{y})
= \int_{-\infty}^{+} \int_{-\infty}^{+} \ddot{u}_n^j \left( \mathbf{x}, t - \tau; \mathbf{y}, 0 \right) P_t \left( \mathbf{y}, \omega \right) e^{i\omega \tau} \, d\tau \, d\Omega(\mathbf{y})
= e^{i\omega \tau} \int_{-\infty}^{+} \int_{-\infty}^{+} \ddot{u}_n^j \left( \mathbf{x}, t; \mathbf{y}, 0 \right) e^{-i\omega \tau} P_t \left( \mathbf{y}, \omega \right) e^{i\omega \tau} \, d\tau \, d\Omega(\mathbf{y})
\]
will appear in front of all integrals, the following frequency-domain version of the integral identity is obtained,

\[ \int_{\Gamma} \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}) \mathcal{I}(\mathbf{y}, \omega) \, d\Gamma(\mathbf{y}) + \int_{\Gamma} 2i\omega \nu_n \eta_n(\mathbf{y}) \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}) U_l(\mathbf{y}, \omega) - \frac{\partial}{\partial y_j} \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}) U_l(\mathbf{y}, \omega) \, d\Gamma(\mathbf{y}) \]

where

\[ \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}) = \int_{-\infty}^{\infty} \tilde{u}_a^b(\mathbf{x} \mp t \mathbf{y}, 0) e^{-i\omega t} \, dt, \]  

(39)

\[ \tilde{P}_a^b(\mathbf{x}, \omega; \mathbf{y}) = \int_{-\infty}^{\infty} \tilde{p}_a^b(\mathbf{x} \mp t \mathbf{y}, 0) e^{-i\omega t} \, dt, \]  

(40)

\[ \tilde{B}_a^b(\mathbf{x}, \omega; \mathbf{y}) = \int_{-\infty}^{\infty} \tilde{b}_a^b(\mathbf{x} \mp t \mathbf{y}, 0) e^{-i\omega t} \, dt. \]  

(41)

These are identified as the Fourier transforms of \( \tilde{u}_a^b(\mathbf{x} \pm t \mathbf{y}, 0) \), \( \tilde{p}_a^b(\mathbf{x} \pm t \mathbf{y}, 0) \), and \( \tilde{b}_a^b(\mathbf{x} \pm t \mathbf{y}, 0) \), respectively.

Combining Eqs. (8) and (41), it becomes evident that, in the frequency domain, the load in the fundamental solution state is applied as \( \rho B_a^b(\mathbf{x}, \omega; \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \delta_d \). Inserting this into Eq. (38) and assuming that the physical field is free of volume forces, i.e. \( B_l(\mathbf{y}, \omega) = 0 \), the following surface integral identity is achieved,

\[ C_a^b(\mathbf{x}) U_l(\mathbf{x}, \omega) + \int_{\Gamma} \tilde{P}_a^b(\mathbf{x}, \omega; \mathbf{y}) U_l(\mathbf{y}, \omega) \, d\Gamma(\mathbf{y}) = \int_{\Gamma} \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}) \mathcal{I}(\mathbf{y}, \omega) \, d\Gamma(\mathbf{y}) + \int_{\Gamma} 2i\omega \nu_n \eta_n(\mathbf{y}) \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}) U_l(\mathbf{y}, \omega) - \frac{\partial}{\partial y_j} \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}) U_l(\mathbf{y}, \omega) \, d\Gamma(\mathbf{y}). \]  

(42)

Here, \( C_a^b(\mathbf{x}) \) is a constant, which only depends on the local geometry at the observation point \( \mathbf{x} \). For a point \( \mathbf{x} \in \Omega \), which is interior to the domain, \( C_a^b(\mathbf{x}) = \delta_d \). For a point \( \mathbf{x} \in \Gamma \) belonging to a smooth part of the boundary, \( C_a^b(\mathbf{x}) = \frac{1}{2} \delta_d \). If the boundary is not smooth at \( \mathbf{x} \), other values of \( C_a^b(\mathbf{x}) \) are achieved. Eq. (42) is identified as a moving-frame-of-reference equivalent of the Somigliana Identity in the frequency domain.

On a part of the surface \( \Gamma \), where the normal vector is perpendicular to the surface, the last two integrals in Eq. (42) do not give any contributions, because \( v_j \eta_n(\mathbf{y}) = 0 \). Hence, if the entire surface is parallel to the direction of convection, the integral identity defined in Eq. (42) is identical to the integral identity for a stationary load in a fixed frame of reference, see e.g. [19]. For any other orientation of the surface, the last integral in Eq. (42) must be taken into consideration. In particular it is noticed that (in contrast to the situation in fixed coordinates) the frequency domain integral identity is not identical to the ‘static load’ integral identity.

However, for any practical purposes, parts of the surface, where the last two integrals in Eq. (42) must be included, will not exist. In order to analyse the stationary elastodynamic response in the moving frame of reference, the requirement would in any case be that the geometry should be two-dimensional in the sense that the cross-section of a model must be constant along the direction of convection. This means that the last integral in Eq. (42) can be omitted in the final formulation.

### 2.2 Frequency-Domain Full-Space Green’s Functions

In Eq. (7), i.e. the convection dominated Navier equation, let the displacement field and the volume load correspond to the time-domain fundamental solution. Fourier transformation of the resulting equation with \( \tau = 0 \) then provides,

\[ (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j} \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}) + \mu \frac{\partial^2}{\partial x_j} \delta(\mathbf{x} - \mathbf{y}) \delta_d \]

\[ = \rho \left( -\omega^2 + 2i\omega \nu_j \frac{\partial}{\partial x_j} + v_j \nu_k \frac{\partial^2}{\partial x_j \partial x_k} \right) \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}). \]  

(43)

Here, \( \tilde{U}_a^b(\mathbf{x}, \omega; \mathbf{y}) \) is the Fourier transform of \( \tilde{u}_a^b(\mathbf{x} \pm t \mathbf{y}, 0) \) as defined by Eq. (39). It should be noticed that the sign on the term corresponding to the first time-derivative of \( \tilde{u}_a^b(\mathbf{x} \pm t \mathbf{y}, 0) \) is different from the sign, which is obtained in the frequency-domain governing equation for the physical state, when harmonic variation is assumed in the form given by Eqs. (39) to (41).

In the fixed frame of reference, the Green’s function for the displacement is usually derived from the governing equation by application of the principle of Helmholtz decomposition and subsequent solution of the resulting scalar wave equations. However, in the convection dominated elastodynamic problem, this approach is indesirable because a closed-form solution cannot be established. This is due to the fact that the differential operators in the governing equation are not spherically symmetrical. Hence, in the moving-frame-of-reference case, the frequency-domain Green’s functions are instead found directly by the Fourier transformation of the time-domain equivalents, that is from Eqs. (39) and (40).
Some of the terms in time-domain Green’s functions contain the Dirac delta function or the time derivative thereof. For a term \( \mathcal{F}_{\alpha}^A(t) \) including a Dirac delta spike in the time-domain fundamental solutions, let a function \( \mathcal{F}_{\alpha}^A(t) \) be defined so that

\[
\mathcal{F}_{\alpha}^A(t) e^{-i\omega t} = \mathcal{F}_{\alpha}^A(t) \delta(t - r/c_A).
\]  

(44)

Introducing the variable substitutions \( t = T_A(s) = S_A(t) \), which have been defined above, the integration that must be carried out in order to obtain the Fourier transformation of one of these terms may be expressed as

\[
\int_{-\infty}^{\infty} \mathcal{F}_{\alpha}^A(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} \mathcal{F}_{\alpha}^A(T_A(s)) T_A'(s) \delta(s) ds
= \int_{-\infty}^{\infty} \mathcal{F}_{\alpha}^A(T_A(0)) T_A'(0) = \mathcal{F}_{\alpha}^A(t_A) t_A'.
\]  

(45)

where prime signifies differentiation with respect to the argument. For convenience, the parameter \( t_A' = T_A'(0) \) has been introduced. Likewise, for a term containing the time derivative of the Dirac delta function, let a function \( \mathcal{F}_{\alpha}^A(t) \) be defined, where

\[
\mathcal{F}_{\alpha}^A(t) e^{-i\omega t} = \mathcal{F}_{\alpha}^A(t) \delta(t - r/c_A).
\]  

(46)

For one of these terms, the variable substitution given in Eq. (21), along with integration by parts, leads to the Fourier transformation

\[
\int_{-\infty}^{\infty} \mathcal{F}_{\alpha}^A(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} \mathcal{F}_{\alpha}^A(T_A(s)) T_A'(s) \delta(s) ds
= - \int_{-\infty}^{\infty} \frac{d}{ds} \left( \mathcal{F}_{\alpha}^A(T_A(s)) T_A'(s) \right) \delta(s) ds
= - \int_{-\infty}^{\infty} \mathcal{F}_{\alpha}^A(T_A(0)) T_A'(0) = \mathcal{F}_{\alpha}^A(t_A) t_A'.
\]  

(47)

where \( \mathcal{F}_{\alpha}^A(t_A) = \frac{d}{dt} \mathcal{F}_{\alpha}^A(t_A) \) and \( t_A' = T_A'(0) \). The first two derivatives of \( T_A(s) \) are

\[
T_A''(s) = \frac{\chi_A(s)}{\chi_A(s)} + \frac{n_j^2}{\chi_A(s)} \frac{n_j^2}{\chi_A(s)} - \frac{n_j^2}{\chi_A(s)} - \frac{n_j^2}{\chi_A(s)}.
\]  

(48)

\[
T_A''(s) = \frac{\chi_A(s)}{\chi_A(s)} + \frac{n_j^2}{\chi_A(s)} \frac{n_j^2}{\chi_A(s)} - \frac{n_j^2}{\chi_A(s)} - \frac{n_j^2}{\chi_A(s)}.
\]  

(49)

With the integration rules defined in Eqs. (45) and (47), a closed-form Fourier transformation can be carried out for most of the terms in the fundamental solutions for both the displacement and the traction. However, the first terms of the Green’s functions cannot be Fourier transformed analytically, and a numerical approach is therefore necessary. A simple numerical integration of the entire integrand, e.g. use of the trapezoidal rule, is inefficient, in particular for high frequencies. Here, the exponential term varies rapidly and a lot of points must therefore be included in the numerical integration. However, a parametric study (no results are given here) indicates that, for all subsonic velocities, the integrands, except for the exponential terms, are smooth and slowly varying functions of time, which may be fitted by polynomials of a reasonably low order. It is noticed that two points are sufficient in order to provide an exact solution for \( \nu = 0 \), because the arrival times of the wavefronts here are linear functions of the distance \( \tau = r \). For subsonic velocities \( \nu \neq 0 \), four to five points are found to give satisfactory results for Mach numbers below approximately 0.9 relative to \( c_s \). However, since the approximation is only carried out for a small part of the entire Green’s functions, the error occurring for Mach numbers closer to 1.0 are small, when 4-5 points are used.

Thus, instead of Fourier transforming the original terms \( \mathcal{F}_{\alpha}^A(t) \) of the time-domain Green’s functions, the following approximation is introduced,

\[
\mathcal{I}_{\alpha} = \int_{t_p}^{t_s} \mathcal{F}_{\alpha}(t) e^{-i\omega t} dt \approx \int_{t_p}^{t_s} \mathcal{L}_{\alpha}(t) e^{-i\omega t} dt.
\]

(50)

Here \( \mathcal{L}_{\alpha}(t) \) is a polynomial of order \( n \). Lagrange interpolation is used. Hence \( \mathcal{L}_{\alpha}(t) \) is constructed from \( n+1 \) points of the original Green’s function terms, including the end-points (i.e. the arrival times \( t_p \) and \( t_s \)).

\[
\mathcal{L}_{\alpha}(t) = \sum_{k=0}^{n} \frac{I_k(t)}{I_k(t_k)} f_k, \quad I_k(t) = \prod_{j=0, j \neq k}^{n} (t_j - t),
\]

(51)

where \( f_k = \mathcal{F}_{\alpha}(t_k) \).

The terms of the Lagrange interpolation function \( \mathcal{L}_{\alpha}(t) \) may be rearranged into a standard polynomial form, \( \mathcal{L}_{\alpha}(t) = p_0 + p_1 t + \ldots + p_n t^n \). The integral of Eq. (50) then becomes

\[
\mathcal{I}_{\alpha} \approx \int_{t_p}^{t_s} \mathcal{L}_{\alpha}(t) e^{-i\omega t} dt
= \left[ e^{i\omega t} \sum_{k=0}^{n} \frac{1}{(\omega t)^{n+1}} \left( \sum_{j=0}^{n} \frac{j!}{(j-k)!} p_j t^{j-k} \right) \right]_{t_p}^{t_s}.
\]

(52)

Let \( r_P^P = r_i(\mathbf{x}, t_P; \mathbf{y}, 0) \), \( r_P = r(\mathbf{x}, t_P; \mathbf{y}, 0) \) and \( r_S^S = r_i(\mathbf{x}, t_S; \mathbf{y}, 0) \), \( r_S = r(\mathbf{x}, t_S; \mathbf{y}, 0) \). Making use of Eqs. (21) and (44) to (52), the Fourier transformations defined in Eqs. (39) and (40) lead to the following frequency-domain Green’s functions for the displacement and the surface traction,

\[
\tilde{U}_{\alpha}^A(\mathbf{x}, \mathbf{y}; \omega) = \frac{1}{4\pi} \left[ \mathcal{I}_{\alpha} + \frac{1}{c_p} \frac{r_P^P}{r_P} \mathcal{I}_{\alpha} e^{-i\omega t_P} + \frac{1}{c_s} \frac{\delta_A}{r_S} - \frac{r_S^S}{r_S} \mathcal{I}_{\alpha} e^{-i\omega t_S} \right],
\]

(53)

\[
\tilde{P}_{\alpha}^A(\mathbf{x}, \mathbf{y}; \omega) = \frac{1}{4\pi} \left[ \mathcal{I}_{\alpha} + (A_d^P + B_d^P + C_d^P) e^{-i\omega t_P} + (A_d^S + B_d^S + C_d^S) e^{-i\omega t_S} \right].
\]

(54)
The integrals \( I_{h}^{G} \) and \( I_{h}^{P} \) occurring in Eqs. (53) and (54), respectively, are given as

\[
I_{h}^{G} = \int_{\Gamma_{p}} s \left( \frac{3}{r_{p}} \frac{r_{p} t_{p}^{2}}{r^{3}} - \frac{\delta_{h} t_{p}^{2}}{r^{3}} \right) \left( \frac{r_{p} t_{p}}{r^{2}} \right) t_{p} \left( 1 - e^{-i\omega t} \right) dt, \\
I_{h}^{P} = \int_{\Gamma_{p}} s \left( \frac{3}{r_{p}} \frac{r_{p} t_{p}^{2}}{r^{3}} - \frac{\delta_{h} t_{p}^{2}}{r^{3}} \right) \left( \frac{r_{p} t_{p}}{r^{2}} \right) t_{p} \left( 1 - e^{-i\omega t} \right) dt,
\]

but an approximation of the integrands is carried out in the Lagrange-interpolation sense as defined in Eq. (50). The remaining terms in the fundamental solution for the traction are:

\[
A_{h}^{G} = \left( \frac{2}{c_{p}} \frac{r_{p} t_{p}^{2}}{r^{3}} + \frac{\delta_{h} t_{p}^{2}}{r^{3}} \right) \left( \frac{r_{p} t_{p}}{r^{2}} \right) t_{p} \left( 1 - e^{-i\omega t} \right) dt,
\]

\[
B_{h}^{G} = \frac{2}{c_{p}^{2}} \left( \frac{r_{p} t_{p}^{2}}{r^{3}} + \frac{\delta_{h} t_{p}^{2}}{r^{3}} \right) \left( \frac{r_{p} t_{p}}{r^{2}} \right) t_{p} \left( 1 - e^{-i\omega t} \right) dt,
\]

\[
C_{h}^{G} = \frac{2}{c_{p}^{2}} \frac{r_{p} t_{p}^{2}}{r^{3}} \left[ \frac{r_{p} t_{p}^{2}}{r^{3}} + \frac{\delta_{h} t_{p}^{2}}{r^{3}} \right] n_{j} \left( t_{p}^{2} \right)^{2},
\]

\[
A_{h}^{P} = \left( \frac{2}{c_{p}} \frac{r_{p} t_{p}^{2}}{r^{3}} + \frac{\delta_{h} t_{p}^{2}}{r^{3}} \right) \left( \frac{r_{p} t_{p}}{r^{2}} \right) t_{p} \left( 1 - e^{-i\omega t} \right) dt,
\]

\[
B_{h}^{P} = \frac{2}{c_{p}^{2}} \left( \frac{r_{p} t_{p}^{2}}{r^{3}} + \frac{\delta_{h} t_{p}^{2}}{r^{3}} \right) \left( \frac{r_{p} t_{p}}{r^{2}} \right) t_{p} \left( 1 - e^{-i\omega t} \right) dt,
\]

\[
C_{h}^{P} = \frac{2}{c_{p}^{2}} \frac{r_{p} t_{p}^{2}}{r^{3}} \left[ \frac{r_{p} t_{p}^{2}}{r^{3}} + \frac{\delta_{h} t_{p}^{2}}{r^{3}} \right] n_{j} \left( t_{p}^{2} \right)^{2},
\]

2.3 Boundary Element Discretization

The displacement and traction field on the surface are discretized into \( N \) nodal values. Likewise the boundary itself is discretized into \( NE \) elements, each having the surface \( \Gamma_{j} \). Let \( U_{j}^{(\omega)} \) and \( P_{j}^{(\omega)} \) be two \( (3N_{j} \times 1) \) vectors storing the nodal displacements and tractions, respectively, for the \( N_{j} \) nodes in element \( j \). Furthermore, defining \( x_{j} \) as the \( (3N_{j} \times 1) \) coordinate vector for the element nodes, the displacement and traction fields over the element surface \( \Gamma_{j} \) may be described in vector form,

\[
U(x, \omega) = \Phi_{j}(x) \cdot U_{j}^{(\omega)},
\]

\[
P(x, \omega) = \Phi_{j}(x) \cdot P_{j}^{(\omega)},
\]

where \( x \) is a point \((x_1, x_2, x_3)\) on the element surface \( \Gamma_{j} \) and \( \Phi_{j}(x) \) is a \((3 \times 3N_{j})\) matrix storing the shape functions for the element. In the present analysis, quadrilateral elements with quadratic interpolation are used, and it is assumed that the entire surface is parallel to the direction of convection. With the definitions in Eqs. (63) and (64), the discretized boundary integral equation, i.e. the BEM formulation of Eq. (42), for a single collocation node with the coordinates \( x_{i} \) obtains the form

\[
C(x_{i}) U_{i}^{(\omega)} + \sum_{j=1}^{NE} \left\{ \int_{\Gamma_{j}} P^{*}(x_{i}, \omega; y) \Phi_{j}(y) d\Gamma_{j} \right\} U_{j}^{(\omega)} = \sum_{j=1}^{NE} \left\{ \int_{\Gamma_{j}} U^{*}(x_{i}, \omega; y) \Phi_{j}(y) d\Gamma_{j} \right\} P_{j}^{(\omega)},
\]

where \( U_{i}^{(\omega)} \) are the complex amplitudes of displacement at \( x_{i} \) and the convective integral terms are neglected. Notice that the integrals over each of the element surfaces \( \Gamma_{j} \) is carried out with respect to the source point coordinates \( y \), not \( x_{i} \), which is a single point.

The \( N \) matrix equations for each of the collocation nodes in the BE domain may be assembled into a single, global matrix equation for the entire domain. When discontinuous tractions are allowed over the common edges of adjacent elements, the global system of equations become,

\[
(C + H) U = GP
\]

The \( C \) matrix stores the \((3 \times 3)\) matrices \( C(x_{i}) \) for each of the observation nodes along the diagonal and is otherwise empty. The values of the integrals over the Green’s functions for the traction and displacement are stored in \( H \) and \( G \), respectively, and \( U \) and \( P \) are vectors with the displacements and tractions at each node and for each degree of freedom. The matrices \( C \) and \( H \) both have the dimensions \((3N \times 3N)\), and \( U \) and \( P \) are \((3N \times 1)\) vectors. \( G \) is a \((N' \times 3N)\) matrix, where \( N' = 3 \sum_{j=1}^{NE} N_{j} \) is the sum of degrees of freedom in each of the elements, keeping the degrees of freedom for multiple coincident element nodes separately.
If discontinuous tractions are not allowed, the system matrices all become square matrices with the dimensions $(3N \times 3N)$, and Eq. (66) is reduced to

$$\mathbf{H} \mathbf{U} = \mathbf{G} \mathbf{P},$$  \hspace{1cm} (67)

where the geometry constants are absorbed into the $\mathbf{H}$ matrix. In the numerical examples section, it is tested whether significant discrepancies arise in the results obtained with such a model when the tractions are in fact discontinuous.

In the case, where discontinuous tractions are not allowed, a coupling between two or more boundary element domains may be carried out first turning each individual BE domain into a macro finite element. For this purpose, the original BE system matrices $\mathbf{G}$ and $\mathbf{H}$ for boundary element domain $i$ are turned into an equivalent stiffness matrix,

$$\mathbf{K}_{i\text{bed}} = \mathbf{T}_{i\text{bed}} \{\mathbf{G}_{i\text{bed}}\}^{-1} \mathbf{H}_{i\text{bed}},$$  \hspace{1cm} (68)

where $\mathbf{T}_{i\text{bed}}$ is a transformation matrix expressing the relationship between nodal forces and the surface tractions applied over the surface of the domain. The stiffness matrices of the domains, as defined by Eq. (68), are subsequently assembled to form a global system of equations for the entire model. In the present method of dealing with multiple BE domains, inversion of the $\mathbf{G}$ matrices is required. This may be a costly affair for domains with many degrees of freedom. Alternatively, a coupling may be established in the original boundary element sense, i.e. in terms of surface tractions. However, this leads to global system matrices with greater dimensions, see [18].

### 2.4 Singularities of the Green’s functions

In the boundary element method, numerical problems arise in the spatial integration over the boundary, because the kernel functions of the system matrices are singular. In particular, the traction Green’s function tensor in elastostatics and -dynamics has a strong singularity of the order $1/r^2$, implying that the integration over $P_{ij}^G(\mathbf{x}, \omega; \mathbf{y})$ can only be carried out in the Cauchy Principal Value sense. A comparison of Eqs. (16) and (21) indicates that,

$$\begin{align*}
\tilde{r} & \to 0 \Rightarrow \left\{ \begin{array}{l}
\tilde{t}_P \to 0 \\
\tilde{t}_S \to 0
\end{array} \right. \Rightarrow \left\{ \begin{array}{l}
\tilde{r}_P \to \tilde{r} \\
\tilde{r}_S \to \tilde{r}
\end{array} \right. \hspace{1cm} (69)
\end{align*}$$

Therefore, the moving-frame-of-reference fundamental solution tensors given by Eqs. (53) and (54) have the same qualitative behaviour as their fixed-frame-of-reference counterparts in the limit as $r \to 0$.

The surface integration over $\tilde{U}_i^S(\mathbf{x}, \omega; \mathbf{y})$, which is only weakly singular, may be carried out by means of a standard Gauss-Legendre quadrature rule, provided that a linear element-wise coordinate transformation is applied, in which the Jacobian at the collocation node coinciding with the source point is equal to zero. The Jacobian of the order $O(r)$ cancels out the singularity of the order $1/r$ [22]. The same approach may be taken in order to evaluate the singular terms of the traction Green’s function. Here, however, a non-linear coordinate transformation is necessary, as the Jacobian must at least be of the order $O(r^2)$ for the singularity to vanish. Doblaré and Gracia [23] proposed a four-order coordinate transformation, which may be used for the numerical evaluation of any integral in the Cauchy Principal Value sense and hence also for the present problem.

Alternatively, in the fixed frame of reference, the rigid body motion technique [22] along with the enclosing elements technique [24] has been used for the dynamic analysis of the elastic half-space. Here, a direct evaluation of the singularities of the Cauchy Principal Value integrals is avoided. Further, the method has the advantage over the non-linear coordinate transformation technique that the geometric constants $\mathbf{C}(\mathbf{x}_i)$ occurring in Eq. (65) are found simultaneously. However, this approach is not well-suited for the present problem, because the strong singularity in the traction Green’s function corresponding to the static or quasistatic fundamental solution is not easily identified.

The procedure for dealing with the strong singularities will not be further addressed here, because only problems involving smooth surfaces are considered in the numerical examples below. On such a part of the surface, the singular terms of the traction Green’s function vanish, since the behaviour of the fundamentals solutions in the limit as $r \to 0$ is identical to the behaviour in the non-moving load case.

### 3. Numerical examples

In this section, the boundary element scheme with the derived Green’s functions is tested. Two examples are given, in which a rectangular harmonically varying load is moving uniformly on the smooth surface of a homogeneous or layered half-space. The results are compared with the semi-analytic solution proposed by Jones et al. [4].

#### 3.1 A moving load on a homogeneous half-space

A homogeneous, viscoelastic half-space is considered. The model has the parameters given in Table 1. These parameters correspond to unsaturated sandy soil with medium stiffness - here given in terms of the Young’s modulus, $E$, and the Poisson Ratio, $\nu$, which are related to the Lamé constants as

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \hspace{1cm} \nu = \frac{\lambda}{2(\lambda + \mu)},$$  \hspace{1cm} (70)

The loss factor, $\eta$, defines a complex Young’s modulus as $E^* = E(1 + i\eta)$, which again results in complex values of the Lamé constants and therefore also the phase velocities $c_p$ and $c_S$. Notice that $\eta$ is constant for all frequencies, well knowing that this results in a non-causal system in the time domain. However, the error due to the chosen damping model is very small as long as the loss factor is of the order 0.1 or below.
1. The semi-analytic solution proposed by Jones et al.

2. The original boundary element scheme defined by Eq. (67).

3. The revised boundary element, or macro finite element, scheme defined by Eq. (66), in which discontinuous surface traction is allowed.

For comparison, three methods are used for the analysis:

1. The semi-analytic solution and the semi-analytic solution and the semi-analytic solution. This counts for both the amplitude and the phase shift and for all convection velocities.

2. The BE schemes provide satisfactory results, even very close to the edge of the model. The reason is likely to be the small impedance mismatch between the analysed half-space and the full-space, which is the basis for the Green’s functions.

The wave propagation on the surface of the half-space, i.e. the Rayleigh waves, is illustrated in Fig. 5. Clearly, for the convection velocity \( v = 0 \text{ m/s} \) the Rayleigh wavefronts form concentric circles. As the velocity gets higher, the distance between the centres of each circular wavefront becomes greater.

### Table 1

<table>
<thead>
<tr>
<th>Layer</th>
<th>( E ) [MPa]</th>
<th>( \nu )</th>
<th>( \rho ) [kg/m^3]</th>
<th>( \eta )</th>
<th>Depth [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>369</td>
<td>0.257</td>
<td>1550</td>
<td>0.10</td>
<td>Half-space</td>
</tr>
</tbody>
</table>

A harmonic, uniformly distributed vertical excitation is applied on the surface of the half-space over an area of \( 3 \times 3 \text{ m}^2 \). The analysis is carried out for the frequency 40 Hz and for the four convection speeds 0, 50, 100 and 150 m/s. All convection velocities are below Mach 0.5 with respect to the phase velocity of the S-waves, which is approximately 320 m/s for the properties listed in Table 1.

For the two BE models produce results that are nearly identical to the semi-analytic solution. This counts for both the amplitude and the phase shift and for all convection velocities.

In the two first methods the load is applied in terms of surface traction over the \( 3 \times 3 \text{ m}^2 \) area with the amplitude 1/9 Pa. In the third case the load is applied as point forces adding up to a total of 1 N and distributed according to the shape functions.

With the given convection and phase velocities, the shortest Rayleigh-wavelength, which may occur behind the load relative to the direction of convection, is 7.0 m. The same wavelength occurs to the side of the load, independently of the convection velocity. In front of the load, the minimum wavelength is 3.3 m. In order to get a sufficiently accurate approximation of the field quantities in the BE solutions, approximately 3 elements must be used per wavelength. Thus, elements with a length of 2 m are used behind and to the side of the loaded area, whereas an element length of 1 m is used in front of, and under, the load. The elements under the load have a width of 1.5 m, so that a total of six elements are located within the loaded area. The mesh, which has been used in the BE solutions, is illustrated in Fig. 3. The symmetry in the \( x_2 \)-direction has been utilized in order to reduce the number of degrees of freedom in the model.

In Fig. 4 the results are plotted for the vertical displacements along the \( x_1 \)-axis (see Fig. 3), i.e. on the surface of the ground along the line passing through the centre of the loaded area and parallel to the direction of convection. Signatures \( \bullet \), \( \circ \) and \( \times \) indicate the results from the analyses carried out with methods 1, 2 and 3, respectively. The following observations can be made:

- The boundary element solutions are nearly identical to the semi-analytic solution. This counts for both the amplitude and the phase shift and for all convection velocities.

- The BE schemes provide satisfactory results, even very close to the edge of the model. The reason is likely to be the small impedance mismatch between the analysed half-space and the full-space, which is the basis for the Green’s functions.

The wave propagation on the surface of the half-space, i.e. the Rayleigh waves, is illustrated in Fig. 5. Clearly, for the convection velocity \( v = 0 \text{ m/s} \) the Rayleigh wavefronts form concentric circles. As the velocity gets higher, the distance between the centres of each circular wavefront becomes greater.

### 3.2 A moving load on a layered half-space

A viscoelastic, horizontal layer of a relatively soft soil and with a depth of 2 m is overlaying a homogeneous half-space of a stiffer material. The properties for the two soil materials are listed in Table 2. An analysis is carried out for the layered half-space using methods 1 and 3 described in the previous subsection, i.e. the semi-analytic solution and the BE solution in terms of macro finite elements. Method 2 is not applicable, since it provides no means for a coupling between two BE domains.

Again, a harmonically varying, vertical excitation is applied on the surface over an area of \( 3 \times 3 \text{ m}^2 \). However, the analyses are performed at the frequency 20 Hz and for the convection velocities 0, 25, 50 and 75 m/s. With this
Fig. 4. Vertical displacement along the $\bar{z}_1$-axis due to a rectangular load moving with various speeds on the surface of a homogeneous half-space. Results obtained with the original BEM, the macro-FE BEM and the semi-analytic solution are plotted with the signatures $\circ$, $\times$ and $\cdots$, respectively.
choice of parameters, the boundary element mesh used for the analysis of the homogeneous half-space can be reused in the present analysis.

The results for the vertical displacements on the surface of the ground along the $x_1$-axis are plotted in Fig. 6. Signatures · and × indicate the results from the analyses carried out with methods 1 and 3, respectively. The following observations can be made:

- The results obtained with the semi-analytical approach and the boundary element scheme in the vicinity of and in front of the loaded area are close to being identical.

- Behind the loaded area, the influence of the edge is much more pronounced than it is in the example with the homogeneous half-space. This is in particular the case for the phase shift. The main reason to this problem is that the impedance mismatch between the modelled layer and the ‘full-space’ that lies beyond the edge is much greater than that between the homogeneous half-space and the ‘full-space’.

- The phase shift is not varying linearly with the distance from the loaded area to either side of the load as it is the case for the homogeneous half-space. This is due to the fact that waves are generated in the layered medium other than the Rayleigh wave, which is the only wave that can be observed on the surface of the homogeneous half-space.

Due to the fact that the influence of the edge is bigger and the wave propagation pattern is more complicated than is the case for a homogeneous half-space, a bigger and more refined mesh has to be used for the analysis of the layer half-space than is necessary for a homogeneous half-space.

4. Conclusions

A boundary element formulation has been given for the analysis of the stationary response of an elastic medium to a source moving with constant subsonic velocity. The analysis is carried out in the frequency domain and in a Cartesian coordinate system following the moving source. This approach is novel in the sense that elastodynamic boundary element analysis in the moving local frame of reference has previously only been performed in the time domain. Frequency domain solutions have been confined to the so-called quasi-two-dimensional boundary element method, where the Cartesian coordinate in the direction of the convection is transformed into the wave number domain.

The Green’s functions are formulated in the time domain for the case of subsonic convection, but it is described how to establish the fundamental solutions if trans- or supersonic convection is to be analysed. The frequency-domain Green’s functions for the displacement and the surface traction are derived by Fourier transformation of the corresponding Green’s functions in the time domain. Parts of the kernel functions are approximated by means of Lagrange interpolation. It has been found that an accurate approximation is achieved with only four points in the interpolation including the arrival times of the P- and S-wave fronts. Close to Mach 1 with respect to the S-wave speed, more points may be needed in the interpolation. However, for convection velocities near $c_S$, the elements must be very small in front of and under the load, because the wave lengths in the moving frame of reference become very small. Hence, in practice the proposed method is anyway inappropriate for the analysis of such problems.

Two different boundary element schemes are tested against a semi-analytical method, where the solution is given in wave number domain. In the first BE scheme, discontinuous surface traction is allowed, whereas in the second scheme, the load is applied in terms of point forces as it is the case in the finite element method. The two BE schemes provide results, which are close to being identical, even within the loaded area. Further, for a homogeneous half-space, the results obtained with the boundary element schemes are in good accordance with the semi-analytic solution, and the edge of the BE mesh has only little influence on the quality of the solution. For a layered half-space, bigger discrepancies between the BE and the semi-analytic solution are observed close to the edge behind the load than are present in the model of the homogeneous half-space. However, near the load and in front of it, the BE scheme provides satisfactory results.

The Green’s functions and the accompanying boundary element scheme derived in the paper forms an alternative to the quasi-two-dimensional boundary element method for the analysis of structures with a two-dimensional geometry and subject to a moving harmonic excitation over a finite area.
Fig. 6. Vertical displacement along the $\hat{x}_1$-axis due to a rectangular load moving with various speeds on the surface of a layered half-space. Results obtained with the macro-FE BEM and the semi-analytic solution are plotted with the signatures $\times$ and $\cdots$, respectively.
Only structures with smooth surfaces have been analysed in the paper, but the theory may easily be generalized to the analysis of structures with an arbitrary geometry.

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References


