The figure shows a vertical, continuous, plane beam $ABCD$ with infinite bending stiffness $EI = \infty$ and constant mass per unit length $\mu$. The beam is infinite rigid against axial deformations. The sub-beams $AB$, $BC$ and $CD$ all have the length $a$. The structure is simply supported at the point $B$, and is supported at the point $D$ by a horizontal linear elastic spring with the spring constant $k$. At the point $C$ a horizontal linear viscous damper with the damper constant $c$ is attached. Only small vibrations in the plane of the structure around a vertical statical state of equilibrium are considered.

**Question 1 (15%)**

Formulate the equation of motion for eigenvibrations of the system.

**Question 2 (5%)**

Determine the undamped angular eigenfrequency and the damping ratio of the structure.
PROBLEM 2

The figure shows a plane frame structure made up of the horizontal beam $AB$ and the vertical beam $BC$ both of the length $a$, rigidly connected at the point $B$. Both beams are massless Bernoulli-Euler beams with the constant bending stiffness $EI$, and are infinite rigid against axial deformations. Beam $AB$ is fixed supported at the point $A$, and at the free end at point $C$ a point mass of the magnitude $m$ is attached. Only small vibrations in the plane of the structure around the statical state of equilibrium are considered.

**Question 1** (15%)  
Formulate the equations of motion describing the horizontal and vertical motions of the point $C$.

**Question 2** (15%)  
Determine the undamped angular eigenfrequencies and the undamped mode shapes of the structure.
The figure shows a horizontal, plane, massless Bernoulli-Euler beam with constant bending stiffness $EI$ and the length $a$. The beam is fixed supported at the point $A$, and is infinitely rigid against axial deformations. At the point $B$ a point mass $M$ and a vertical, linear elastic spring with the spring constant $k$ are attached. At the other end of the spring a point mass of the magnitude $m$ is attached, which is constrained so it can only move in the plane of the beam and in the vertical direction. The structure is loaded at point $B$ with a harmonically varying vertical force $P(t) = P_0 \cos(\omega t)$ with amplitude $P_0$ and angular frequency $\omega$. Only small vibrations in the plane of the structure around the statical state of equilibrium are considered.

**Question 1 (15%)**

Determine the equations of motion of the structure.

**Question 2 (10%)**

Determine the mass $m$ so that the point $B$ is at rest, when the response from the initial conditions has been dissipated.
The figure shows a vertical, plane Bernoulli-Euler beam $AB$ of the length $a$. The beam has constant bending stiffness $EI$, constant mass per unit length $\mu$, and is infinitely stiff against axial deformations. The structure is fixed supported at the point $A$. The horizontal displacement of a point in the distance $x$ from the support point in the plane of the beam is denoted $u(x, t)$. The continuous displacement field is approximated by the following two-degrees-of-freedom model

$$u(x, t) \simeq \Phi_1(x) q_1(t) + \Phi_2(x) q_2(t)$$

where

$$\Phi_1(x) = \left(\frac{x}{a}\right)^2, \quad \Phi_2(x) = \left(\frac{x}{a}\right)^3$$

**Question 1 (5%)**

Are the indicated shape functions $\Phi_1(x)$ and $\Phi_2(x)$ valid? Motivate the answer.

**Question 2 (10%)**

Determine the consistent kinetic and potential energy of the structure based on the indicated two-degrees-of-freedom model, and specify the related mass- and stiffness matrices.

**Question 3 (10%)**

Calculate the undamped angular eigenfrequencies based on the two-degrees-of-freedom model. Are the calculated eigenfrequencies upper or lower bounds to the exact solutions? Motivate the answer.
SOLUTIONS

PROBLEM 1

Question 1:

Fig. 1: Loads on free beam and definition of degree of freedom.

Since the beam is infinite rigid in bending and axial deformations and simple supported at point B, it has but a single degree of freedom, which is selected as the rotation \( \theta(t) \) with the sign defined in Fig. 1. Next, the beam is cut free from the spring and the damper, and the spring force \( k \cdot 2a\theta \) and the damper force \( c \cdot a\dot{\theta} \) are applied as external forces with signs as shown in Fig. 1. Further, the inertial moment \( -J\ddot{\theta} \) is applied as an external moment acting in the direction of \( \theta(t) \) according to d’Alembert’s principle. \( J \) denotes the mass moment of inertia of the beam around the point B. Moment equilibrium around the point B then provides the equation of motion

\[
-J\ddot{\theta} - a \cdot c\dot{\theta} - 2a \cdot 2k\theta = 0 \quad \Rightarrow \\
J\ddot{\theta} + ca^2\dot{\theta} + 4ka^2\theta = 0
\]

(1)

where

\[
J = \frac{1}{3}\mu a^3 + \frac{1}{3}\mu (2a)^3 = 3\mu a^3
\]

(2)
Question 2:

From (1), (2) and (2-7), (2-39) follows that undamped angular eigenfrequency $\omega_0$ and the damping ratio $\zeta$ becomes

$$\omega_0 = \sqrt{\frac{4a^2k}{J}} = \sqrt{\frac{4}{3}} \sqrt{\frac{k}{\mu a}} \quad (3)$$

$$\zeta = \frac{ca^2}{2\sqrt{4ka^2 \cdot J}} = \frac{\sqrt{3}}{12} \frac{c}{\sqrt{ka\mu}} \quad (4)$$

PROBLEM 2

Question 1:

The system has two degrees of freedom $x_1(t)$ and $x_2(t)$, which describes the horizontal and vertical displacements of the point mass $m$ from the static equilibrium state with the signs defined in Fig. 1.
The equations of undamped eigenvibration read, cf. (3-3)

\[ \mathbf{x} = \mathbf{D} (-\mathbf{M} \ddot{\mathbf{x}}) \tag{1} \]

where

\[ \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \tag{2} \]

Using the principle of virtual forces the flexibility coefficients become, see Fig. 2

\[
\begin{align*}
\delta_{11} &= \int \frac{M_1^2(s)}{EI} \, ds = \frac{1}{3EI} a^3 + \frac{a^3}{EI} = \frac{4}{3EI} a^3 \\
\delta_{12} = \delta_{21} &= \int \frac{M_1(s)M_2(s)}{EI} \, ds = -\frac{1}{2EI} a^3 \\
\delta_{22} &= \int \frac{M_2^2(s)}{EI} \, ds = \frac{1}{3EI} a^3
\end{align*} \tag{3}
\]

Introducing the stiffness matrix \( \mathbf{K} = \mathbf{D}^{-1} \) the equation of motion become, cf. (3-40)

\[ \mathbf{M} \ddot{\mathbf{x}} + \mathbf{K} \mathbf{x} = 0 \tag{4} \]
where

\[
K = \frac{6 \, EI}{7 \, a^3} \begin{bmatrix} 2 & 3 \\ 3 & 8 \end{bmatrix}
\]  

(5)

**Question 2:**

The undamped angular eigenfrequencies \(\omega_j\) and related eigenmodes \(\Phi^{(j)} = [\Phi_1^{(j)}, \Phi_2^{(j)}]\) are determined from the generalized eigenvalue problem, cf. (3-42)

\[
\begin{bmatrix} 2 - \lambda_j & 3 \\ 3 & 8 - \lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad j = 1, 2
\]

(6)

where

\[
\lambda_j = \frac{7 \, ma^3}{6 \, EI \, \omega_j^2}
\]

(7)

The characteristic equation becomes, cf. (3-43)

\[
\lambda_j^2 - 10\lambda_j + 7 = 0 \quad \Rightarrow \\
\lambda_j = \begin{cases} 5 - \sqrt{18}, & j = 1 \\ 5 + \sqrt{18}, & j = 2 \end{cases}
\]

\[
\omega_j^2 = \begin{cases} \frac{6}{7} \left(5 - \sqrt{18}\right) \frac{EI}{ma^3}, & j = 1 \\ \frac{6}{7} \left(5 + \sqrt{18}\right) \frac{EI}{ma^3}, & j = 2 \end{cases}
\]

(8)

The eigenmodes are normalized as follows

\[
\Phi^{(j)} = \begin{bmatrix} \Phi_1^{(j)} \\ 1 \end{bmatrix}
\]

(9)

The first component is determined from the first equation of (6). Hence
\[3 \cdot \Phi^{(j)} + (8 - \lambda_j) \cdot 1 = 0 \quad \Rightarrow \]

\[\Phi^{(j)} = \begin{cases} 
-1 - \sqrt{2}, & j = 1 \\
-1 + \sqrt{2}, & j = 2 
\end{cases} \]

(10)

Fig. 3: a) 1st eigenmode. b) 2nd eigenmode.

The eigenmodes with the normalization defined in (9) have been sketched in Fig. 3.
PROBLEM 3

Question 1:

![Diagram](image)

Fig. 1: Definition of degrees of freedom. Forces on the point masses $M$ and $m$.

The beam $AB$ is massless. Then, the system has two degrees of freedom $x_1(t)$ and $x_2(t)$, which describes the vertical displacements of the masses $M$ and $m$ from the static equilibrium state with the signs defined in Fig. 1. The spring is cut free from the beam and the point mass $m$. The elongation of the spring becomes $x_2 - x_1$, and the spring force $k(x_2 - x_1)$ is applied on the beam and the mass $m$ with the signs defined in Fig. 1. Additionally, using d’Alembert’s principle the inertial load $-M\ddot{x}_1$ is applied on the mass $M$ along with the external force $P(t)$. Then, the equation of motion of the free beam becomes, cf. (3-1)

$$x_1 = \delta_{11} \left( P(t) - M\ddot{x}_1 + k(x_2 - x_1) \right) \Rightarrow$$

$$M\ddot{x}_1 + Kx_1 + k(x_1 - x_2) = P(t) \tag{1}$$

where

$$K = \frac{1}{\delta_{11}} = \frac{3EI}{a^3} \tag{2}$$

Newton’s 2nd law for the free mass $m$ becomes

$$m\ddot{x}_2 = -k(x_2 - x_1) \Rightarrow$$

$$m\ddot{x}_2 + k(x_2 - x_1) = 0 \tag{3}$$

(1) and (3) may be written in the following matrix form
\[ M\ddot{x} + Kx = f(t) \]  \hspace{1cm} (4)

where

\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \hspace{0.5cm} f(t) = \begin{bmatrix} P(t) \\ 0 \end{bmatrix}, \hspace{0.5cm} M = \begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix}, \hspace{0.5cm} K = \begin{bmatrix} K + k & -k \\ -k & k \end{bmatrix} \]  \hspace{1cm} (5)

**Question 2:**

Since the system is free of damping, the stationary motion is either in phase or counter phase to the harmonic varying excitation \( P(t) = P_0 \cos(\omega t) \). Hence, the stationary solution to (4), when the initial value response is dissipated, is given as

\[ x(t) = X \cos(\omega t) \]  \hspace{1cm} (6)

where, cf. (3-101), (3-102)

\[ X = H(\omega)F \]  \hspace{1cm} (7)

\[ F = \begin{bmatrix} P_0 \\ 0 \end{bmatrix} \]  \hspace{1cm} (8)

\[ H(\omega) = \left( -\omega^2 M + K \right)^{-1} = \frac{1}{D} \begin{bmatrix} k - \omega^2 m & k \\ k & K + k - \omega^2 M \end{bmatrix} \]  \hspace{1cm} (9)

\[ D = (K + k - \omega^2 M)(k - \omega^2 m) - k^2 \]  \hspace{1cm} (10)

The amplitude of the displacement of the point \( B \) becomes

\[ X_1 = \frac{1}{D} \left( k - \omega^2 m \right) P_0 \]  \hspace{1cm} (11)

Hence, the point \( B \) is at rest if the mass \( m \) is chosen as

\[ m = \frac{k}{\omega^2} \]  \hspace{1cm} (12)
**PROBLEM 4**

**Question 1:**
The geometrical boundary conditions reads

$$ u \bigg|_{x=0} = \frac{\partial u}{\partial x} \bigg|_{x=0} = 0 $$

(1)

Both shape functions $\Phi_1$ and $\Phi_2$ fulfill these conditions, and hence are legal shape functions.

**Question 2:**
The kinetic energy becomes

$$ T(t) = \frac{1}{2} \int_0^a \mu u^2(x, t) \, dx = \frac{1}{2} \int_0^a \mu \left( \Phi_1(x)q_1(t) + \Phi_2(x)q_2(t) \right)^2 \, dx = \frac{1}{2} \mathbf{q}^T(t) \mathbf{M} \mathbf{q}(t) $$

(2)

where

$$ \mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} $$

(3)

$\mathbf{M}$ is the consistent mass matrix given as

$$ \mathbf{M} = \int_0^a \mu \begin{bmatrix} \Phi_1^2 & \Phi_1 \Phi_2 \\ \Phi_1 \Phi_2 & \Phi_2^2 \end{bmatrix} \, dx = \frac{\mu a}{210} \begin{bmatrix} 42 & 35 \\ 35 & 30 \end{bmatrix} $$

(4)

The potential energy becomes

$$ U(t) = \frac{1}{2} \int_0^a EI \left( \frac{\partial u^2(x, t)}{\partial x^2} \right) \, dx = \frac{1}{2} \int_0^a \mu \left( \Phi_1''(x)q_1(t) + \Phi_2''(x)q_2(t) \right)^2 \, dx = \frac{1}{2} \mathbf{q}^T(t) \mathbf{K} \mathbf{q}(t) $$

(5)

The stiffness matrix becomes

$$ \mathbf{K} = \int_0^a EI \begin{bmatrix} \Phi_1''^2 & \Phi_1'' \Phi_2'' \\ \Phi_1'' \Phi_2'' & \Phi_2''^2 \end{bmatrix} \, dx = \frac{EI}{a^3} \begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix} $$

(6)
Question 3:

The generalized eigenvalue problem becomes, cf. (3-42)

\[
\begin{bmatrix}
4 - 42\lambda_j & 6 - 35\lambda_j \\
6 - 35\lambda_j & 12 - 30\lambda_j
\end{bmatrix}
\begin{bmatrix}
\Phi_1^{(j)} \\
\Phi_2^{(j)}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad j = 1, 2
\]

(7)

where

\[
\lambda_j = \frac{1}{210} \frac{\mu a^4}{EI} \omega_j^2
\]

(8)

The characteristic equation becomes, cf. (3-43)

\[
35\lambda_j^2 - 204\lambda_j + 12 = 0 \quad \Rightarrow
\]

\[
\lambda_j = \begin{cases} 
0.059429, & j = 1 \\
5.769142, & j = 2
\end{cases}
\]

\[
\omega_j = \begin{cases} 
3.532732 \sqrt{\frac{EI}{\mu a^4}}, & j = 1 \\
34.806893 \sqrt{\frac{EI}{\mu a^4}}, & j = 2
\end{cases}
\]

(9)

The exact solutions become, cf. (4-41)

\[
\omega_j = \begin{cases} 
3.516015 \sqrt{\frac{EI}{\mu a^4}}, & j = 1 \\
22.034491 \sqrt{\frac{EI}{\mu a^4}}, & j = 2
\end{cases}
\]

(10)

The numerical solutions are upper bounds to the analytical solutions. This is a consequence of the continuous generalization of Rayleigh’s fraction, valid for all shape functions fulfilling the geometrical boundary conditions.