EIGENVIBRATIONS AND FORCED VIBRATIONS OF LINEARLY VISCOUS SDOF OSCILLATOR

Fig. 1: Ramp force \( f(t) \).

The equation of motion of a linearly viscous damped SDOF oscillator reads, cf. (2-32), (2-33)

\[
m \left( \ddot{x} + 2\zeta \omega_0 \dot{x} + \omega_0^2 x \right) = f(t) \quad t > 0
\]

\[
x(0) = x_0 \quad \dot{x}(0) = \dot{x}_0
\]

where \( m \) is the mass, \( \zeta \) is the damping ratio, cf. (2-39), and \( \omega_0 \) is the undamped circular eigenfrequency. The ramp force on the system is given as, see fig. 1

\[
f(t) = \begin{cases} f_0 \frac{t}{\Delta t} & t \in [0, \Delta t] \\ 0 & t \in (\Delta t, \infty) \end{cases}
\]  

(2)

Formally, \( f(t) \) and its time-derivative, \( f'(t) \), can be written

\[
f(t) = \frac{f_0}{\Delta t} t U(\Delta t - t) \quad f'(t) = \frac{f_0}{\Delta t} \left( U(\Delta t - t) - t \delta(\Delta t - t) \right)
\]  

(3)

where \( U(t) \) is the Heaviside’s unit step function defined as

\[
U(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}
\]  

(4)

The delta-spike at \( t = \Delta t \) in \( f'(t) \) occurs, because \( f(t) \) is discontinuous with a jump of magnitude \(-f_0\) at this instant of time, see figs. 1 and 2. Dirac’s delta function may formally be interpreted as the time derivative of Heaviside’s unit step function, i.e.

\[
\delta(t) = \frac{d}{dt} U(t)
\]  

(5)
The solution to (1) reads, cf. (2-121)
\[
x(t) = e^{-\zeta \omega_0 t} \left( x_0 \cos \left( \sqrt{1 - \zeta^2} \omega_0 t \right) + \frac{x_0 + \zeta \omega_0 x_0}{\omega_0 \sqrt{1 - \zeta^2}} \sin \left( \sqrt{1 - \zeta^2} \omega_0 t \right) \right) + \\
\int_0^t h(t - \tau) f(\tau) d\tau
\]

(6)

\( h(t) \) signifies the impulse response function as given by (2-110). For the ramp load (2) the Duhamel's integral in (6) becomes
\[
\int_0^t h(t - \tau) f(\tau) d\tau = \int_0^t h(t - \tau) f_0 \frac{\tau}{\Delta t} d\tau
\]

(7)

\( t_0 = \min (t, \Delta t) \)

(8)

The impulse response function, \( h(t) \), is given by, cf. (2-110)
\[
h(t) = \begin{cases} 
0 & , \quad t < 0 \\
\frac{1}{m \omega_0 \sqrt{1 - \zeta^2}} e^{-\zeta \omega_0 t} \sin \left( \omega_0 \sqrt{1 - \zeta^2} t \right) & , \quad t \geq 0
\end{cases}
\]

(9)

The integral (7) is evaluated by partial integration
\[
\int_0^t h(t - \tau) f(\tau) d\tau = - \left[ H(t - \tau) f(\tau) \right]_0^t + \int_0^t H(t - \tau) f'(\tau) d\tau
\]

(10)

where
\[
H(t) = \int_0^t h(\tau) d\tau
\]

(11)

Insertion of (3) into (10) and use of \( H(0) = 0, \quad f(0) = 0 \) provides
\[
\int_0^t h(t - \tau) f(\tau) d\tau = \int_0^t H(t - \tau) f_0 \frac{\tau}{\Delta t} \left( U(\Delta t - \tau) - \tau \delta(\Delta t - \tau) \right) d\tau = \\
\frac{f_0}{\Delta t} \int_0^{\Delta t} H(t - \tau) d\tau - f_0 H(t - \Delta t)
\]

(12)

In order to evaluate the integral in (12) the initial value problem (2-109) for \( h(t) \) is considered
\[
h(t) = \begin{cases} 
\ddot{h}(\tau) + 2\zeta \omega_0 \dot{h}(\tau) + \omega_0^2 h(\tau) = 0 \\
h(0) = 0, \quad \dot{h}(0) = \frac{1}{m}
\end{cases}
\]

(13)
Integration of (13) over the interval $[0, t]$ provides the following expression of $H(t)$ in terms of $h(t)$ and $\dot{h}(t)$

$$\left[ \dot{h}(\tau) + 2\zeta \omega_0 h(\tau) + \omega_0^2 H(\tau) \right]_0^t = 0 \quad \Rightarrow$$

$$H(t) = \frac{1}{\omega_0^2} \left( \frac{1}{m} - \dot{h}(t) - 2\zeta \omega_0 h(t) \right)$$

(14)

or

$$H(t) = \frac{1}{m \omega_d \omega_0} \left( \omega_d - e^{-\zeta \omega_0 t} \left( \zeta \omega_0 \sin(\omega_d t) + \omega_d \cos(\omega_d t) \right) \right)$$

(15)

where $\omega_d = \omega_0 \sqrt{1 - \zeta^2}$ is the damped circular eigenfrequency, cf. (2-49). Further, the initial conditions in (13) and $H(0) = 0$ have been used. (14) is inserted into (12), which provides

$$\int_0^t h(t - \tau) f(\tau) d\tau = \frac{f_0}{\Delta t \omega_0^2} \int_0^t \left( \frac{1}{m} - \dot{h}(t - \tau) - 2\zeta \omega_0 h(t - \tau) \right) d\tau - f_0 H(t - \Delta t) =$$

$$\frac{f_0}{\Delta t \omega_0^2} \left[ \frac{t}{m} + h(t - \tau) + 2\zeta \omega_0 H(t - \tau) \right]_0^t - f_0 H(t - \Delta t) =$$

$$\frac{f_0}{\Delta t \omega_0^2} \left( \frac{t_0}{m} + h(t - t_0) - h(t) + 2\zeta \omega_0 \left( H(t - t_0) - H(t) \right) \right) - f_0 H(t - \Delta t)$$

(16)

Insertion of (9) and (15) into (16) provides the final expression for the Duhamel's integral. Notice, that $H(t) \to \frac{1}{m \omega_0^2}$ and $\dot{h}(t) \to 0$ as $t \to \infty$. Hence, the integral in (16) has the asymptotic value $\frac{f_0}{\Delta t \omega_0^2} \frac{\Delta t}{m} - \frac{f_0}{m \omega_0} = 0$, as expected for a load with finite duration acting on a damped system.