Outline of Lecture 1

- Introduction.
  - Harmonic and Periodic Motions.
  - Equation of Motion of Undamped Eigenvibrations.
  - Equation of Motion for Forced, Damped Vibrations.
  - Damping Models.
  - Analytical Dynamics.
Introduction

Vibration analysis:

Determination of the displacement and the internal forces of a structure due to time-dependent external loads or initial conditions.

Description of motion:

Time-dependent coordinates:

\[ \mathbf{x}^T(t) = [x_1(t), \ldots, x_n(t)] \]  

\[ x_i(t) \quad : \text{Degree of freedom.} \]

Specifies the displacement components and rotation components of a system of mass particles or extended rigid bodies from a known referential state (static equilibrium state).
Discrete and continuous systems:

- Single-degree-of-freedom (SDOF) system: $n = 1$
- Multi-degree-of-freedom (MDOF) system: $1 < n < \infty$
- Continuous system: $n = \infty$

SDOF and MDOF systems:

Discrete systems.

Linear and nonlinear systems:

Equations of motions are formulated in terms of linear or nonlinear ordinary differential equations (discrete systems), or linear or nonlinear partial differential equations (continuous systems).
Harmonic and Periodic Motions

$x(t) = A \cos(\omega t - \Psi)$  

- $A$ : Amplitude.
- $\omega$ : Angular frequency, $[s^{-1}]$.
- $\Psi$ : Phase angle.
Period $T$ defined from:

$$x(t + T) = A \cos \left( \omega(t + T) - \Psi \right) = A \cos(\omega t - \Psi) = x(t)$$  \hspace{1cm} (3)

Hence

$$\omega T = 2\pi \quad \Rightarrow$$

$$T = \frac{2\pi}{\omega} \hspace{1cm} (4)$$

- $f = \frac{1}{T}$: Frequency, [Hz]. Number of vibration periods per unit of time.
Complex calculus:

Euler's equation:

\[ e^{i\theta} = \cos \theta + i \sin \theta \]  
\[ e^{i\theta} = \cos \theta + i \sin \theta \]
\[ e^{-i\theta} = \cos \theta - i \sin \theta \] \[ \Rightarrow \]

\[ \cos \theta = \frac{1}{2} \left( e^{i\theta} + e^{-i\theta} \right) \]  
\[ \sin \theta = \frac{1}{2i} \left( e^{i\theta} - e^{-i\theta} \right) \]
Polar representation of complex numbers:

\[
z = x + iy = Ae^{i\theta} = A \cos \theta + iA \sin \theta
\]

\[
x = A \cos \theta \quad \Rightarrow \quad A = \sqrt{x^2 + y^2}
\]
\[
y = A \sin \theta \quad \Rightarrow \quad \theta = \arctan \frac{y}{x}
\]

- \(A\) : Modulus (magnitude).
- \(\theta\) : Argument.

Multiplication and division of complex numbers:

\[
z_1 \cdot z_2 = A_1 e^{i\theta_1} \cdot A_2 e^{i\theta_2} = A_1 A_2 e^{i(\theta_1 + \theta_2)}
\]

\[
\frac{z_1}{z_2} = \frac{A_1 e^{i\theta_1}}{A_2 e^{i\theta_2}} = \frac{A_1}{A_2} e^{i(\theta_1 - \theta_2)}
\]
Complex conjugation:

\[ z^* = (x + iy)^* = x - iy = A \cos \theta - i A \sin \theta = \]
\[ A \left( \cos(-\theta) + i \sin(-\theta) \right) = A e^{-i\theta} \] (11)

\[ (z_1 z_2)^* = \left( A_1 A_2 e^{i(\theta_1 + \theta_2)} \right)^* = A_1 A_2 e^{-i(\theta_1 + \theta_2)} = \]
\[ A_1 e^{-i\theta_1} A_2 e^{-i\theta_2} = z_1^* z_2^* \] (12)

\[ \left( \frac{z_1}{z_2} \right)^* = \left( \frac{A_1}{A_2} e^{i(\theta_1 - \theta_2)} \right)^* = \frac{A_1}{A_2} e^{-i(\theta_1 - \theta_2)} = \frac{A_1 e^{-i\theta_1}}{A_2 e^{-i\theta_2}} = \frac{z_1^*}{z_2^*} \] (13)

Complex conjugation of a product or a fraction is performed by complex conjugating each factor of the numerator and of the denominator separately. Especially,

\[ z \cdot z^* = Ae^{i\theta} \cdot Ae^{-i\theta} = A^2 \] (14)
Example 1: Calculation of products and fractions of complex numbers

Let:

\[ z_1 = 0.25 - 0.5i = 0.5570e^{-1.1071i} \]
\[ z_2 = -0.10 + 0.75i = 0.7566e^{1.7033i} \]

Calculate:

\[ z = \frac{z_1^4}{z_2^2} = 0.1707e^{-1.5518i} = 0.0032 - 0.1705i \]
\[ z = \frac{(z_1^*)^4}{z_2^2} = 0.1706e^{+1.0219i} = 0.0890 + 0.1455i \]
Complex representation of a harmonic motion:

\[ x(t) = A \cos (\omega t - \Psi) = \text{Re} \left( A \cos (\omega t - \Psi) + iA \sin (\omega t - \Psi) \right) = \text{Re} \left( A e^{i(\omega t - \Psi)} \right) = \text{Re} \left( A e^{-i\Psi} e^{i\omega t} \right) = \text{Re} \left( B e^{i\omega t} \right) \] 

- \( B = A e^{-i\Psi} \): Complex amplitude. \( A = |B| \)

\( B \) contains information of both the amplitude \( A \) (modulus of \( B \)) and the phase angle \( \Psi \) (negative argument of \( B \)).
Periodic motion:

The motion repeats itself after the time interval $T$:

$$x(t) = x(t + T)$$  \hspace{1cm} (16)

- $T$: Period.

Figure 2: Periodic motion.
Fourier series representation of a periodic motion:

\[
x(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left( a_j \cos(\omega_j t) + b_j \sin(\omega_j t) \right)
\]

\[
\omega_j = n \cdot \frac{2\pi}{T} = n\omega_1 \quad , \quad j = 1, 2, \ldots
\]

\[
\begin{align*}
a_j &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(\omega_j t) \, dt \quad , \quad j = 0, 1, \ldots \\
b_j &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(\omega_j t) \, dt \quad , \quad j = 1, 2, \ldots
\end{align*}
\]

- \[
\frac{a_0}{2} = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, dt \quad : \text{Mean (time averaged) motion.}
\]

The Fourier series converges towards \( x(t) \) in continuity points and towards \( \frac{1}{2} (x(t^+) + x(t^-)) \) in discontinuity points.
Representation of the Fourier series as an infinite sum of harmonic motions:

\[ x(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} c_j \cos(\omega_j t - \Psi_j) \]

\[ c_j \cos(\omega_j t - \Psi_j) = c_j \cos \Psi_j \cos(\omega_j t) + c_j \sin \Psi_j \sin(\omega_j t) \quad \Rightarrow \]

\[
\begin{align*}
  a_j &= c_j \cos \Psi_j \\
  b_j &= c_j \sin \Psi_j \\
  c_j &= \sqrt{a_j^2 + b_j^2} \\
  \Psi_j &= \arctan \frac{b_j}{a_j}
\end{align*}
\]
Complex representation of the Fourier series of a periodic motion:

\[ x(t) = \sum_{j=-\infty}^{\infty} A_j e^{i\omega_j t}, \quad \omega_j = j \frac{2\pi}{T}, \quad j = 0, \pm 1, \pm 2, \ldots \]  \hspace{1cm} (21)

\[ A_j = A_{-j}^* = \begin{cases} 
\frac{1}{2}(a_j - ib_j) & , \quad j = 1, 2, \ldots \\
\frac{1}{2}a_0 & , \quad j = 0 \\
\frac{1}{2}(a_{-j} + ib_{-j}) & , \quad j = -1, -2, \ldots 
\end{cases} \hspace{1cm} (22) \]
Proof:

Use of $\omega_{-j} = -\omega_j$ and $A_{-j} = A_j^*$ provides

$$x(t) = \frac{a_0}{2} + \sum_{j=-\infty}^{-1} A_j e^{i\omega_j t} + \sum_{j=1}^{\infty} A_j e^{i\omega_j t}$$

$$= \frac{a_0}{2} + \sum_{j=1}^{\infty} \left( A_j^* e^{-i\omega_j t} + A_j e^{i\omega_j t} \right)$$

$$= \frac{a_0}{2} + \sum_{j=1}^{\infty} \left( \frac{1}{2} (a_j + ib_j) e^{-i\omega_j t} + \frac{1}{2} (a_j - ib_j) e^{i\omega_j t} \right)$$

$$= \frac{a_0}{2} + \sum_{j=1}^{\infty} \left( a_j \cdot \frac{1}{2} (e^{i\omega_j t} + e^{-i\omega_j t}) + b_j \cdot \frac{1}{2i} (e^{i\omega_j t} - e^{-i\omega_j t}) \right)$$

$$= \frac{a_0}{2} + \sum_{j=1}^{\infty} \left( a_j \cos(\omega_j t) + b_j \sin(\omega_j t) \right) \quad (23)$$
Example 2: Fourier series expansion of a periodic triangular signal

The Fourier series, the equivalent sum of harmonic motions and the complex representation of the triangular signal shown on Fig. 3 are determined. The function value is wanted at $t = \frac{1}{2}T, \frac{3}{2}T, \ldots$ at any finite truncation of the series (17), and for infinite many harmonic components.

Figure 3: Triangular signal.
The Fourier coefficients become:

\[ a_j = \begin{cases} 
  x_0 & , \quad j = 0 \\
  0 & , \quad j = 1, 2, \ldots 
\end{cases} \]
\[ b_j = -\frac{x_0}{\pi j} , \quad j = 1, 2, \ldots \] \hspace{1cm} (24)

\[ x(t) = \frac{1}{2} x_0 - \sum_{j=1}^{\infty} \frac{x_0}{j\pi} \sin(\omega_j t) \]
\[ = \frac{1}{2} x_0 - \sum_{j=1}^{\infty} \frac{x_0}{j\pi} \cos\left(\omega_j t - \frac{\pi}{2}\right) \]
\[ = \frac{1}{2} x_0 + \sum_{j=-\infty}^{-1} \frac{i}{2} b_{-j} e^{i\omega_j t} - \sum_{j=1}^{\infty} \frac{i}{2} b_j e^{i\omega_j t} , \quad \omega_j = j \frac{2\pi}{T} \] \hspace{1cm} (25)

Since, \( \sin \left(\omega_j \frac{T}{2}\right) = \sin \left(j \pi\right) = 0 \) it follows that the exact value \( x \left( \frac{T}{2} \right) = \frac{x_0}{2} \) is obtained, no matter how many terms are retained in the series.
Single-Degree-of-Freedom Systems

- \( m \): Mass of particle, [kg].
- \( k \): Spring constant of linear elastic spring, [N/m].

Figure 4: Eigenvibrations of a linear undamped SDOF system. a) Undeformed state. b) Static equilibrium state. c) Initial state. d) Dynamic deformed state. e) Free mass loaded with external and internal forces.
Only motion in the vertical direction is considered. One coordinate (degree of freedom) is necessary.

a) Spring in the undeformed state.

b) Spring with the mass in the static equilibrium state due to gravity. Elongation of the spring: \( x_s = \frac{mg}{k} \).

c) Initial displacement \( x_0 \) and velocity \( \dot{x}_0 \) of the mass at the time \( t = 0 \). Measured from the static equilibrium state.

d) Undamped eigenvibrations of mass, \( t > 0 \). The elongation of the spring becomes \( x(t) + x_s \), assuming that \( x(t) \) is measured from the static equilibrium state.
General principle for formulating equations of motion:

Valid for SDOF, MDOF and continuous systems.

1) All masses are cut free.
2) All external forces (in casu $mg$) and internal forces (in casu $k(x + x_s)$) are applied on the free masses as defined on Fig. 4e. External loads and moments are considered positive in the same direction as the degrees of freedom.
3) Newton’s 2nd law of motion is applied for all free masses.

Equation of motion for the mass particle:

$$m\frac{d^2}{dt^2}(x(t) + x_s) = mg - k(x + x_s) \Rightarrow$$

$$m\ddot{x} + kx = 0 \quad (mg = kx_s) \quad (26)$$
General principle:

Static forces (in casu the gravity force) disappear from the equations of motion, if

1) The motions are measured from the static equilibrium state.
2) The system is linear. Non-linear systems are usually measured from the undeformed state.

The velocity $\dot{x}(t)$ and the acceleration $\ddot{x}(t)$ are considered positive in the same direction as $x(t)$.

Example 3: Equation of motion measured from the undeformed state

$$x_1(t) = x(t) + x_s \Rightarrow$$

$$m\ddot{x}_1 + kx_1 = mg \quad (27)$$
Initial value problem for SDOF system:

\[
\begin{align*}
\ddot{x} + \omega_0^2 x &= 0 \quad , \quad t > 0 \\
x(0) &= x_0 \quad , \quad \dot{x}(0) = \dot{x}_0
\end{align*}
\] (28)

- \( \omega_0 = \sqrt{\frac{k}{m}} \) : Angular eigenfrequency, [s\(^{-1}\)]. (29)
- \( f_0 = \frac{\omega_0}{2\pi} \) : Eigenfrequency, [Hz]. (30)
- \( T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0} \) : Eigenvibration period, [s]. (31)
Box 1: Solution theory for homogeneous linear differential equations of 2\textsuperscript{nd} order

\[ m\ddot{x} + c\dot{x} + kx = 0 \]  \hspace{1cm} (32)

Complete solution:

\[ x_c(t) = a_1 x_1(t) + a_2 x_2(t) \]  \hspace{1cm} (33)

where \( a_1 \) and \( a_2 \) are arbitrary constants, and \( x_1(t) \) and \( x_2(t) \) are linear independent solutions to (32), i.e. \( a_1 x_1(t) + a_2 x_2(t) \neq 0 \). \( a_1 \) and \( a_2 \) are determined from the initial values.

Solution to (28):

\[ x(t) = x_0 \cos(\omega_0 t) + \frac{\dot{x}_0}{\omega_0} \sin(\omega_0 t) \]  \hspace{1cm} (34)
Example 4: Eigenfrequency of a flywheel

Equation of moment of momentum:

$$\frac{dN}{dt} = \frac{d}{dt}(J\dot{\theta}) = -M \left( = \sum \text{all external moments in the direction of } \theta \right) \quad (35)$$

- $N = J\dot{\theta}$: Moment of momentum, [$kgm^2/s$].
- $J$: Mass moment of inertia of flywheel, [$kg \cdot m^2$].
- $M = \frac{GK}{l}\theta$: St. Venant torsional moment in bar, [Nm].
- $G$: Shear modulus, [N/m²].
- $K = \frac{\pi}{32}D^4$: Torsional constant of circular cylindrical bar, [m⁴].
- $D, l$: Diameter and length of bar, [m].

Figure 5: Torsional eigenvibrations of a flywheel.
Structural Dynamics
Lecture 1

\[ \ddot{\theta} + \omega_0^2 \theta = 0 \quad (37) \]
\[ \omega_0^2 = \frac{GK}{Jl} = \frac{\pi}{32} \frac{GD^4}{Jl} \quad (38) \]

**Box 2**: Newton’s 2\(^{nd}\) law of motion and the equation of moment of momentum for a particle system

- \( O \): Arbitrary selected fixed referential point.
- \( \mathbf{r}_i(t) \): Position vector for mass particle \( m_i \).
- \( \mathbf{f}_i(t) \): External force vector on particle \( m_i \).
- \( \mathbf{f}_{ij}(t) \): Internal force vector on particle \( m_i \) from particle \( m_j \).

\[ (\mathbf{f}_{ij}(t) = -\mathbf{f}_{ji}(t), \text{Newton’s 3}\(^{rd}\) law). \]

**Figure 6**: System of 2 particles.
Newton’s 2\textsuperscript{nd} law of motion for particles $m_1$ and $m_2$:

\[
\begin{align*}
    m_1 \ddot{r}_1 &= f_1(t) + f_{12}(t) \\
    m_2 \ddot{r}_2 &= f_2(t) + f_{21}(t) = f_2(t) - f_{12}(t)
\end{align*}
\] (39)

Addition of equations:

\[
    m_1 \ddot{r}_1 + m_2 \ddot{r}_2 = \frac{d}{dt} (m_1 \dot{r}_1 + m_2 \dot{r}_2) = f_1(t) + f_2(t)
\] (40)

Generalization to $n$ particles:

\[
    \frac{d}{dt} M(t) = \sum_{j=1}^{n} f_j(t)
\] (41)

- $M(t) = \sum_{j=1}^{n} m_j \dot{r}_j(t)$ : Momentum vector of the particle system. (42)
Newton 2\textsuperscript{nd} law for a particle system:
The rate of the momentum vector of the particle system is equal to the vector sum of all external forces on the various particles. Internal forces cancel mutually.

The 1\textsuperscript{st} and 2\textsuperscript{nd} equations in (39) are multiplied vectorially with $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, followed by an addition of the result:

$$
\mathbf{r}_1 \times m_1 \ddot{\mathbf{r}}_1 + \mathbf{r}_2 \times m_2 \ddot{\mathbf{r}}_2 = \mathbf{r}_1 \times \mathbf{f}_1 + \mathbf{r}_1 \times \mathbf{f}_{12} + \mathbf{r}_2 \times \mathbf{f}_2 - \mathbf{r}_2 \times \mathbf{f}_{12}
$$

$$
\mathbf{r}_1 \times \mathbf{f}_1 + \mathbf{r}_2 \times \mathbf{f}_2 + (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{f}_{12}
$$

Now:

$$
\mathbf{r}_j \times m_j \ddot{\mathbf{r}}_j = \frac{d}{dt} (\mathbf{r}_j \times m_j \dot{\mathbf{r}}_j) - \dot{\mathbf{r}}_j \times m_j \dot{\mathbf{r}}_j = \frac{d}{dt} (\mathbf{r}_j \times m_j \dot{\mathbf{r}}_j)
$$

$$
(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{f}_{12} = 0
$$

(44)
\[ \dot{r}_j \times m_j \ddot{r}_j = 0 \quad \text{and} \quad (r_1 - r_2) \times f_{12} = 0, \] 
because \( \dot{r}_j \) is parallel to \( m_j \ddot{r}_j \), and because \( (r_1 - r_2) \) is parallel to \( f_{12} \).

Then, (43) reduces to

\[ \frac{d}{dt} (r_1 \times m_1 \dot{r}_1 + r_2 \times m_2 \dot{r}_2) = r_1 \times f_1 + r_2 \times f_2 \quad (45) \]

**Generalization to \( n \) particles:**

\[ \frac{d}{dt} N(t) = \sum_{j=1}^{n} r_j(t) \times f_j(t) \quad (46) \]

- \[ N(t) = \sum_{j=1}^{n} r_j(t) \times m_j \dot{r}_j(t) \quad : \text{Moment of momentum vector} \]

  of the particle system. \quad (47)
Equation of moment of momentum:
The rate of the vector sum of the moment of momentum of all mass particles around an arbitrary fixed point is equal to the vector sum of the moment of all external forces on the various particles around the same point. The moment contribution from all internal forces cancel mutually.
Example 5: Moment of momentum and mass moment of inertia of a rotating circular cylindrical flywheel

Figure 7: Mass moment of inertia of a circular cylindrical flywheel.
Moment of momentum of all particles \( dm = \rho r \, d\theta \, dr \, t \):

\[
N = \int_0^R \int_0^{2\pi} r \cdot (\rho r \, d\theta \, dr \, t) \, r \, \dot{\theta} = 2\pi \rho t \dot{\theta} \int_0^R r^3 \, dr = \frac{\pi}{2} \rho R^4 t \dot{\theta} \quad \Rightarrow
\]

\[
J = \frac{\pi}{2} \rho R^4 t = \frac{1}{2} m R^2 \quad , \quad [\text{kgm}^2]
\]

- \( \rho \) : Mass density, [kg/m\(^3\)].
- \( R, t \) : Radius and thickness of flywheel, [m].
- \( m = \rho \pi R^2 t \) : Mass of flywheel, [kg].
Example 6: Eigenvibrations of a mathematical pendulum

Consider a mathematical pendulum with the length $l$ and the mass $m$, performing free vibrations under the action of gravity.

Determine the non-linear equation of motion for undamped eigenvibrations around a vertical static equilibrium state.

Figure 8: Undamped eigenvibrations of a mathematical pendulum.
Next, linearize the equation of motion assuming that $|\theta| \ll 1$, and determine the angular eigenfrequency, eigenfrequency and eigenvibration period.

Moment of momentum equation around the support point:

$$N = l \cdot \left( m l \dot{\theta} \right) \Rightarrow$$

$$m l^2 \ddot{\theta} = -m g l \sin \theta \quad \Rightarrow$$

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \quad \Rightarrow$$

$$\ddot{\theta} + \frac{g}{l} \theta \simeq 0 \quad (50)$$

$$\omega_0 = \sqrt{\frac{g}{l}} \quad , \quad f_0 = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \quad , \quad T_0 = 2\pi \sqrt{\frac{l}{g}} \quad (51)$$

- $g$ : Acceleration of gravity, [m/s$^2$].
Equation of Motion for Forced, Damped Vibrations

The system described by (28) never stops. For this reason a damping element to extract mechanical energy is introduced parallel to a linear elastic spring free of dissipation. The damping element models dissipation inside the actual spring.

- \( x(t) \): Displacement from the static equilibrium state.
- \( f(t) \): External dynamic load (additional to possible static loads).
  - Positive in direction of \( \dot{x} \).
- \( f_d(t) \): Damping force. Positive in opposite direction of \( \dot{x} \).
  - (co-directional to the spring force \( kx \)).

Figure 9: Forced vibrations of a damped SDOF system.
Newton’s 2\textsuperscript{nd} law for the free mass particle:

\[ m\ddot{x} = f(t) - kx - f_d(t) \quad \Rightarrow \]
\[ m\ddot{x} + kx = f(t) - f_d(t) \quad (52) \]

Power balance of the SDOF dynamical system:

\[ \dot{x} (m\ddot{x} + kx) = \dot{x}f - \dot{x}f_d \quad \Rightarrow \]
\[ \frac{d}{dt} \left( \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 \right) = f\dot{x} - f_d\dot{x} \quad \Rightarrow \]
\[ \frac{d}{dt} E(t) = P(t) - P_d(t) \quad (53) \]
- $E(t) = T(t) + U(t)$: Mechanical energy, [J].
- $T(t) = \frac{1}{2} m \dot{x}^2(t)$: Kinetic energy, [J].
- $U(t) = \frac{1}{2} k x^2(t)$: Strain energy, [J].
- $P(t) = f(t) \dot{x}(t)$: Supplied power of the external force, [W].
- $P_d(t) = f_d(t) \dot{x}(t)$: Dissipated power of internal damping force, [W].

Notice that if $f(t)$ is a moment, then $x(t)$ is a rotation.

**Power balance of a SDOF, MDOF and continuous dynamical system:**
The rate of the mechanical energy of the system is equal to the supplied power from all external dynamic forces minus the dissipated power from all internal damping forces.
Damping Models

Figure 10: Hysteretic loops for the damping force during a harmonic motion. a) Linear viscous damping. b) Fluid damping. c) Coulomb damping (dry friction).

Dissipative damping forces:

\[ f_d \dot{x} > 0 \quad , \quad \dot{x} \neq 0 \quad (54) \]

Examples of dissipative damping forces:

\[
\begin{align*}
 f_d &= c \dot{x} \quad , \quad c > 0 \quad : \text{Linear viscous damping.} \\
 f_d &= c \dot{x} |\dot{x}| \quad , \quad c > 0 \quad : \text{Fluid damping (non-linear).} \\
 f_d &= c \frac{\dot{x}}{|\dot{x}|} \quad , \quad c > 0 \quad : \text{Coulomb damping (non-linear).}
\end{align*}
\]
Equivalent linear viscous damping coefficient of a nonlinear damping model during harmonic motion:

Given a harmonic motion \( x(t) = A \cos(\omega t) \Rightarrow \dot{x}(t) = -A\omega \sin(\omega t) \).

Identical dissipation of energy per vibration period \( T = \frac{2\pi}{\omega} \) requires an equivalent linear viscous damping coefficient \( c_e \) given as:

\[
\int_0^T (c_e \dot{x}) \, dx = \int_0^T f_d \, dx = \int_0^T f_d(t) \dot{x}(t) \, dt \Rightarrow \\
\]

\[
c_e = \frac{\int_0^T f_d(t) \dot{x}(t) \, dt}{\int_0^T \dot{x}^2(t) \, dt} = \frac{-A\omega \int_0^T f_d(t) \sin(\omega t) \, dt}{(A\omega)^2 \int_0^T \sin^2(\omega t) \, dt} = \\
- \frac{1}{A\omega \cdot \frac{T}{2}} \int_0^T f_d(t) \sin(\omega t) \, dt = -\frac{1}{\pi A} \int_0^T f_d(t) \sin(\omega t) \, dt \tag{56}
\]
Example 7: Equivalent linear viscous damping coefficients for fluid and Coulomb damping

\[ c_e = -\frac{1}{\pi A} \int_0^T c\hat{x}|\hat{x}| \sin(\omega t) \, dt = \]

\[ \frac{A\omega^2}{\pi} c \int_0^T \sin^2(\omega t) |\sin(\omega t)| \, dt = \frac{8}{3\pi} A\omega c \]  
(57)

\[ c_e = -\frac{1}{\pi A} \int_0^T c\hat{x} \sin(\omega t) \, dt = \]

\[ \frac{1}{\pi A} c \int_0^T |\sin(\omega t)| \, dt = \frac{4}{\pi} \frac{c}{A\omega} \]  
(58)

In both cases \( c_e \) depends on the vibration amplitude \( A \) and on the angular vibration frequency \( \omega \) via the product \( A\omega \).
Linear Viscous Damped System:

The damping force $f_d = c \dot{x}$ is transferred to the left-hand side:

$$m \ddot{x} + c \dot{x} + kx = f(t), \quad t > 0$$
$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$$

(59)

**Box 3:** Solution theory for non-homogeneous linear differential equations of 2nd order

$$x(t) = x_c(t) + x_p(t) = a_1 x_1(t) + a_2 x_2(t) + x_p(t)$$

(60)

$x_p(t)$ is an arbitrary particular solution to the non-homogeneous differential equation, and $x_c(t)$ is the so-called complementary solution, i.e. the complete solution to the homogeneous differential equation (where $f(t) \equiv 0$) as given by (33). $a_1$ and $a_2$ are determined from the initial values.
Box 3 (cont.)

In dynamics $x_c(t)$ is called an *eigenvibration*, and $x_p(t)$ is denoted the *stationary motion*, i.e. the part of the motion, which sustains, when the eigenvibrations from the initial condition has been dissipated.

- **Analytical Dynamics**
  - $f(t)$: Conservative dynamic load, [N].
  - $T(x) = \frac{1}{2}m\dot{x}^2$: Kinetic energy, [J].
  - $U(x) = \frac{1}{2}kx^2 - f(t)x(t)$: Potential energy (strain energy plus potential energy of $f(t)$), [J].
Introduce the Lagrange function \( L(x, \dot{x}) \) ("Lagrangian") defined as

\[
L(x, \dot{x}) = T(\dot{x}) - U(x)
\]  \hspace{1cm} (61)

Then, the equation of motion (52) is obtained from Lagrange’s equation as:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = f_{nc}(t)
\]  \hspace{1cm} (62)

where \( f_{nc}(t) = -f_d(t) \) is the non-conservative (i.e. the dissipative) part of the load on the free particle.

Analytical dynamics is the preferred method for formulating equations of motion in systems with substructures performing rigid body motions (wind turbine rotors, gears). Internal reaction forces between moving rigid substructures are automatically eliminated.
Proof:

From (61) follows

\[
\begin{align*}
\frac{\partial L}{\partial \dot{x}} &= \frac{\partial T(\dot{x})}{\partial x} = \frac{d}{dx} \left( \frac{1}{2} m \dot{x}^2 \right) = m\ddot{x} \\
\frac{\partial L}{\partial x} &= -\frac{\partial U(x)}{\partial x} = -\frac{d}{dx} \left( \frac{1}{2} k x^2 - f(t)x \right) = -kx + f(t)
\end{align*}
\]

\[
\Rightarrow \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = m\ddot{x} + kx - f(t) = f_{nc}(t) = -c\dot{x} \quad \Rightarrow
\]

\[
m\ddot{x} + c\dot{x} + kx = f(t)
\]  \hspace{1cm} (63)
Summary of Lecture 1

- Harmonic and Periodic Motions.
  Definition of angular frequency, frequency and vibration period.
- Complex Representation of Harmonic Motions.
- Fourier Series Representation of Periodic Motions.
  - General principles for formulating equations of motion.
  - Undamped eigenvibrations.
  - Newton’s 2nd law and equation of moment of momentum for a particle system.
  - Equation of motion for forced, damped vibrations.
  - Damping models: Linear viscous damping, fluid damping, Coulomb damping. Equivalent viscous damping.
  - Formulation of equations of motion by analytical dynamics. Internal forces between moving rigid bodies disappear.