Outline of Lecture 10

- Continuous Systems (cont.).
  - Dynamic Modelling of Structures.
    - SDOF Model.
    - 2DOF Model.
  - Introduction to the Finite Element Method.
    - Equations of Motion of a Plane Bernoulli-Euler Beam Element.
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Continuous Systems (cont.)

Dynamic Modelling of Structures

Dynamic modelling means the reduction of an infinite many degrees-of-freedom system to a system with finite many degrees of freedom.

The partial differential equation with given boundary values of the problem is replaced by an ordinary matrix differential equation:

\[
M \ddot{q} + C \dot{q} + K q = F(t) \quad , \quad t > 0
\]
\[
q(0) = q_0 \quad , \quad \dot{q}(0) = \dot{q}_0
\]  

- \( q(t) \) : Vector of selected degrees-of-freedom vector, \( n \times 1 \).
- \( M \) : Mass matrix, \( n \times n \).
- \( C \) : Damping matrix, \( n \times n \).
- \( K \) : Stiffness matrix, \( n \times n \).
- \( F(t) \) : Dynamic load vector, \( n \times 1 \).
The modelling involves the specification of \( q(t), M, C, K, F(t) \), so the discrete system describes the continuous system “at best”.

This is done by interpolating the continuous displacement field of the structure by a discrete number of appropriate shape functions with a corresponding number of amplitude functions of time, \( q_i(t) \), which represent the selected degrees of freedom. The optimal behaviour of \( q_i(t) \) follows from analytical dynamics. \( C \) is determined by Rayleigh or Caughey damping, presuming that a given number of modal damping ratios are measured or prescribed.

The method will be illustrated with respect to the wind turbine blade shown on Fig. 1, which is modelled as a cantilever Bernoulli-Euler beam.
Figure 1: Rotating wind turbine blade.
The boundary value problem of a rotating blade reads, cf. Lecture 9 Eq. (39) and Lecture 9, Fig. 9b:

\[
\frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) - \frac{\partial}{\partial x} \left( N(x) \frac{\partial u}{\partial x} \right) + \mu(x) \frac{\partial^2 u}{\partial t^2} = p(x, t)
\]

\[
u(0, t) = \frac{\partial u(0, t)}{\partial x} = 0
\]

\[
EI(l) \frac{\partial^3 u(l, t)}{\partial x^3} = 0, \quad EI(l) \frac{\partial^3 u(l, t)}{\partial x^3} - N(l) \frac{\partial u(l, t)}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial^3 u(l, t)}{\partial x^3} = 0
\]

- \( \Omega \): Rotational speed of blade \([s^{-1}]\).
- \( U \): Mean wind velocity \([m/s]\).
- \( \mu(x) \): Mass per unit length, \([kg/m]\).
- \( EI(x) \): Bending stiffness around the \(y\)-axis, \([Nm^2]\).
- \( N(x) \): Axial force due to centrifugal acceleration, \([N]\). \(N(t) = 0\).
- \( u(x, t) \): Displacement in the \(y\)-direction (out-of-rotor plane), \([m]\).
- \( p(x, t) \): Dynamic load per unit length in the \(y\)-direction, \([N/m]\).
Axial force (centrifugal force):

\[ N(x) = \Omega^2 \int_x^l y \mu(y) dy \]  

(3)

Interpolation of the displacement field relative to the rigid body motion of the blade:

\[ u(x, t) \simeq \Phi_1(x)q_1(t) + \Phi_2(x)q_2(t) + \cdots + \Phi_n(x)q_n(t) = \Phi^T(x)q(t) \]  

(4)

\[ q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} \]  

(5)
\[
\Phi(x) = \begin{bmatrix}
\Phi_1(x) \\
\Phi_2(x) \\
\vdots \\
\Phi_n(x)
\end{bmatrix}
\]

- \( \Phi_i(x) \): Shape function (interpolation function).
- \( q_i(t) \): Generalized coordinates (degrees of freedom).

The shape functions must be linearly independent, and must fulfill the geometric boundary conditions at the hub:

\[
\Phi_i(0) = \frac{d\Phi_i(0)}{dx} = 0
\]
The shape functions need not fulfill the mechanical boundary conditions at $x = l \left( \Phi''(l) = \Phi'''(l) = 0 \right)$. It is favourable that one or both of these boundary conditions are fulfilled. This makes the undamped eigenmodes $\Phi^{(j)}(x)$ the best choice for the shape functions.

**Kinetic energy of the continuous system:**

$$T_c = \int_0^l \frac{1}{2} \mu(x) \, dx \, \dot{u}^2(x, t)$$  \hspace{1cm} (8)

**Potential energy of the continuous system:**

$$U_c = \int_0^l \left( \frac{1}{2} EI(x) \left( \frac{\partial^2 u(x, t)}{\partial x^2} \right)^2 + \frac{1}{2} N(x) \left( \frac{\partial u(x, t)}{\partial x} \right)^2 \right) \, dx$$

$$- \int_0^l u(x, t)p(x, t) \, dx$$  \hspace{1cm} (9)
Kinetic and potential energy of the discrete system:

\[ T_d(\dot{q}) = \frac{1}{2} \int_0^l \mu(x)\dot{u}^T(x,t)\dot{u}(x,t)dx = \]

\[ \frac{1}{2} \int_0^l \mu(x)\dot{q}^T(t)\Phi(x)\Phi^T(x)\dot{q}(t)dx = \]

\[ \frac{1}{2} \dot{q}^T(t) \left( \int_0^l \mu(x)\Phi(x)\Phi^T(x)dx \right) \dot{q}(t) = \]

\[ \frac{1}{2} \dot{q}^T(t)M\dot{q}(t) \quad (10) \]

\[ M = \int_0^l \mu(x)\Phi(x)\Phi^T(x)dx \quad : \text{Consistent mass matrix.} \quad (11) \]

Generally, the consistent mass matrix is not diagonal.
\[ U_d(q) = \int_0^l \left( \frac{1}{2} EI(x)q^T(t) \frac{d^2 \Phi(x)}{dx^2} \left( \frac{d^2 \Phi(x)}{dx^2} \right)^T q(t) \right) + \]

\[ \frac{1}{2} N(x)q^T(t) \frac{d\Phi(x)}{dx} \left( \frac{d\Phi(x)}{dx} \right)^T q(t) \right) dx - \int_0^l q^T(t) \Phi(x)p(x,t) dx = \]

\[ \frac{1}{2} q^T(t) \int_0^l \left( EI(x) \frac{d^2 \Phi(x)}{dx^2} \left( \frac{d^2 \Phi(x)}{dx^2} \right)^T + N(x) \frac{d\Phi(x)}{dx} \left( \frac{d\Phi(x)}{dx} \right)^T \right) dx q(t) \]

\[ - q^T(t) \int_0^l \Phi(x)p(x,t) dx = \]

\[ \frac{1}{2} q^T(l)Kq(l) - q^T(l)F(l) \] (12)

\[ K = K_e + K_g \] (13)
\[ K_e = \int_0^l EI(x) \frac{d^2 \Phi(x)}{dx^2} \left( \frac{d^2 \Phi(x)}{dx^2} \right)^T \, dx \]  

(14)

\[ K_g = \int_0^l N(x) \frac{d\Phi(x)}{dx} \left( \frac{d\Phi(x)}{dx} \right)^T \, dx \]  

(15)

\[ F(t) = \int_0^l \Phi(x)p(x, t) \, dx \]  

(16)

- \( K \): Total stiffness matrix.
- \( K_e \): Elastic stiffness matrix.
- \( K_g \): Geometric stiffness matrix.
- \( F(t) \): Dynamic load vector.
“Best” approximation:

\[ T_d = T_c \quad , \quad U_d = U_c \quad \quad \quad (17) \]

(17) implies that \( \mathbf{M}, \mathbf{K} \) and \( \mathbf{F}(t) \) are calculated from (11), (13-15) and (16).

The Lagrangian becomes, cf. Lecture 4, Eq. (10):

\[ L(q, \dot{q}) = T_d(\dot{q}) - U_d(q) = \frac{1}{2} \dot{q}^T \mathbf{M} \dot{q} - \frac{1}{2} q^T \mathbf{K} q + q^T \mathbf{F}(t) \quad \quad \quad (18) \]

Lagrange’s equations of motion, cf. Lecture 4, Eq. (13):

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^T} \right) - \frac{\partial L}{\partial q^T} = \mathbf{F}_{nc}(t) = -\mathbf{C} \dot{q} \quad \Rightarrow \]

\[ \mathbf{M} \ddot{q} + \mathbf{C} \dot{q} + \mathbf{K} q = \mathbf{F}(t) \quad \quad \quad (19) \]
The damping matrix $C$ must be constructed so that any available modal damping ratios are reproduced.

- **SDOF Model**

Figure 2: Shape function of SDOF model.
The displacement field is approximated as:

\[ u(x, t) \simeq \Phi(x)q(t) \] (20)

The shape function is taken as:

\[ \Phi(x) = \frac{3}{2} \left( \frac{x}{l} \right)^2 \left( 1 - \frac{1}{3} \frac{x}{l} \right) \] (21)

Since, \( \Phi(l) = 1 \) the generalized coordinate \( q(t) \) may be interpreted as the tip displacement of the blade.
The derivatives of the shape function become:

\[
\begin{align*}
\Phi'(x) &= \frac{3}{l} \frac{x}{l} \left( 1 - \frac{1}{2} \frac{x}{l} \right) \\
\Phi''(x) &= \frac{3}{l^2} \left( 1 - \frac{x}{l} \right) \\
\Phi'''(x) &= -\frac{3}{l^3} x
\end{align*}
\]  

(22)

The kinematic boundary conditions $\Phi(0) = \Phi'(0) = 0$ and the mechanical boundary condition $\Phi''(l) = 0$ (vanishing bending moment at the tip) are fulfilled. The mechanical boundary condition $\Phi'''(l) = 0$ (vanishing shear force) is not fulfilled. The chosen shape function is considered acceptable.

The mass, the stiffness and the dynamic load of the SDOF model becomes, cf. Eqs. (11), (13-15) and (16):
\[ m = \int_0^l \mu(x) \Phi'^2(x) \, dx = \frac{9}{4} \int_0^l \mu(x) \left( \frac{x}{l} \right)^4 \left( 1 - \frac{1}{3} \frac{x}{l} \right)^2 \, dx \]  \hspace{1cm} (23)

\[ k = k_e + k_g \]  \hspace{1cm} (24)

\[ k_e = \int_0^l EI(x) (\Phi''(x))^2 \, dx = \frac{9}{l^4} \int_0^l EI(x) \left( 1 - \frac{x}{l} \right)^2 \, dx \]  \hspace{1cm} (25)

\[ k_g = \int_0^l N(x) (\Phi'(x))^2 \, dx = \frac{9}{l^2} \int_0^l N(x) \left( \frac{x}{l} \right)^2 \left( 1 - \frac{1}{2} \frac{x}{l} \right)^2 \, dx \]  \hspace{1cm} (26)

\[ f(t) = \int_0^l \Phi(x) p(x, t) \, dx \]  \hspace{1cm} (27)
Example 1: Rotating cantilever beam with constant cross-section

![Diagram of a rotating cantilever beam with constant cross-section]

Figure 3: Rotating cantilever beam with constant cross-section.

The axial force becomes, cf. (3)

\[ N(x) = \Omega^2 \int_x^l y \mu \, dy = \frac{1}{2} \mu \Omega^2 (l^2 - x^2) \]  

(28)
The mass and stiffnesses become, cf. (23-26):

\[
m = \frac{9}{4} \mu \int_0^l \left( \frac{x}{l} \right)^4 \left( 1 - \frac{1}{3} \frac{x}{l} \right)^2 \, dx = \frac{33}{140} \mu l
\]

\[
k_e = \frac{9}{l^4} EI \int_0^l \left( 1 - \frac{x}{l} \right)^2 \, dx = \frac{3}{l^3} \frac{EI}{l^3}
\]

\[
k_g = \frac{9}{l^2} \cdot \frac{1}{2} \mu \Omega^2 l^2 \int_0^l \left( 1 - \left( \frac{x}{l} \right)^2 \right) \left( \frac{x}{l} \right)^2 \left( 1 - \frac{1}{2} \frac{x}{l} \right)^2 \, dx = \frac{81}{280} \mu l \Omega^2
\]

The angular eigenfrequency at \( \Omega = 0 \) becomes:

\[
\omega_{1,0} = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{140}{11}} \sqrt{\frac{EI}{\mu l^4}} = 3.5675 \sqrt{\frac{EI}{\mu l^4}}
\]
The exact values become, cf. Lecture 9, Eq. (69):

\[
\omega_{n,0} = \begin{cases} 
1.8751^2 \sqrt{\frac{EI}{\mu l^4}} = 3.5160 \sqrt{\frac{EI}{\mu l^4}}, & n = 1 \\
4.6941^2 \sqrt{\frac{EI}{\mu l^4}} = 22.035 \sqrt{\frac{EI}{\mu l^4}}, & n = 2
\end{cases}
\]

(31)

The numerical estimate (30) is an upper-bound due to Rayleigh’s principle.

The angular eigenfrequency at \( \Omega > 0 \) becomes:

\[
\omega_1 = \sqrt{\frac{k_e}{m} + \frac{k_g}{m}} = \omega_{1,0} \sqrt{1 + \frac{k_g}{k_e}} = \omega_{1,0} \sqrt{1 + \frac{27}{22} \frac{\Omega^2}{\omega_{1,0}^2}}
\]

(32)

The increase in the angular eigenfrequency depends on the fraction \( \Omega^2/\omega_{1,0}^2 \). The factor \( \frac{27}{22} \) is rather accurate, because the numerical estimates of \( k_e \) and \( k_g \) are both upper-bounds, so the fraction \( k_g/k_e \) is somewhat balanced.
2DOF Model

The displacement field is approximated as:

\[ u(x, t) \simeq \Phi_1(x)q_1(t) + \Phi_2(x)q_2(t) \]  (33)
The shape functions are taken as:

\[
\begin{align*}
\Phi_1(x) &= \left(\frac{x}{l}\right)^2 \left(3 - 2\frac{x}{l}\right) \\
\Phi_2(x) &= \left(\frac{x}{l}\right)^2 \left(1 - \frac{x}{l}\right) l
\end{align*}
\]

It follows from Fig. 4 that the coordinates \( q_1(t) \) and \( q_2(t) \) may be interpreted as the displacement \( u(l,t) \) and the rotation \( \frac{\partial u(l,t)}{\partial x} \) of the tip.

The derivatives of the shape functions become:

\[
\begin{align*}
\Phi'_1(x) &= \frac{6}{l} \frac{x}{l} \left(1 - \frac{x}{l}\right) \\
\Phi''_1(x) &= \frac{6}{l^2} \left(1 - 2\frac{x}{l}\right)
\end{align*}
\]
The kinematic boundary conditions \( \Phi_1(0) = \Phi_1'(0) = 0 \) and \( \Phi_2(0) = \Phi_2'(0) = 0 \) are fulfilled. None of the shape functions fulfill the mechanical boundary conditions \( \Phi_1''(l) = \Phi_1'''(l) = 0 \) and \( \Phi_2''(l) = \Phi_2'''(l) = 0 \).

The mass- and the stiffness matrices and the load vector of the 2DOF model become, cf. Eqs. (11), (13-15) and (16):

\[
\begin{align*}
\Phi'_2(x) &= \frac{x}{l} \left( 2 - 3 \frac{x}{l} \right) \\
\Phi''_2(x) &= \frac{2}{l} \left( 1 - 3 \frac{x}{l} \right)
\end{align*}
\]
\[ M = \int_0^l \mu(x) \begin{bmatrix} \Phi_1^2(x) & \Phi_1(x)\Phi_2(x) \\ \Phi_1(x)\Phi_2(x) & \Phi_2^2(x) \end{bmatrix} dx = \]

\[ \int_0^l \mu(x) \begin{bmatrix} (\frac{x}{l})^4 (3 - 2 \frac{x}{l})^2 & (\frac{x}{l})^4 \left( 3 - 5 \frac{x}{l} + 2 \left( \frac{x}{l} \right)^2 \right) l \\ (\frac{x}{l})^4 \left( 3 - 5 \frac{x}{l} + 2 \left( \frac{x}{l} \right)^2 \right) l & (\frac{x}{l})^4 \left( 1 - \frac{x}{l} \right)^2 l^2 \end{bmatrix} dx \] (37)

\[ K_e = \int_0^l EI(x) \begin{bmatrix} \Phi_1''(x)^2 & \Phi_1'(x)\Phi_2''(x) \\ \Phi_1'(x)\Phi_2''(x) & \Phi_2''(x)^2 \end{bmatrix} dx = \]

\[ \int_0^l EI(x) \begin{bmatrix} 36 \left( 1 - 2 \frac{x}{l} \right)^2 & 12 \left( 1 - 5 \frac{x}{l} + 6 \left( \frac{x}{l} \right)^2 \right) l \\ 12 \left( 1 - 5 \frac{x}{l} + 6 \left( \frac{x}{l} \right)^2 \right) l & 4 \left( 1 - 3 \frac{x}{l} \right)^2 l^2 \end{bmatrix} dx \] (38)
\[ K_g = \int_0^l N(x) \begin{bmatrix} (\Phi'_1(x))^2 & \Phi'_1(x)\Phi'_2(x) \\ \Phi'_1(x)\Phi'_2(x) & (\Phi'_2(x))^2 \end{bmatrix} dx = \]

\[ \int_0^l \frac{N(x)}{l^2} \begin{bmatrix} \frac{36}{l^2} (\frac{x}{l})^2 \left(1 - \frac{x}{l}\right)^2 & \frac{6}{l^2} \left(2 - 5\frac{x}{l} + 3 \left(\frac{x}{l}\right)^2\right) l \\ \frac{6}{l^2} \left(2 - 5\frac{x}{l} + 3 \left(\frac{x}{l}\right)^2\right) l & \left(\frac{x}{l}\right)^2 \left(2 - 3\frac{x}{l}\right)^2 l^2 \end{bmatrix} dx \]

(39)

\[ F(t) = \int_0^l \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \end{bmatrix} p(x, t) dx = \int_0^l \begin{bmatrix} \left(\frac{x}{l}\right)^2 \left(3 - 2\frac{x}{l}\right) \\ \left(\frac{x}{l}\right)^2 \left(1 - \frac{x}{l}\right) l \end{bmatrix} p(x, t) dx \]

(40)
Example 2: 2DOF model of a rotating cantilever beam

The cantilever beam shown in Fig. 3 is considered again using a 2DOF model with the shape function given by (34).

The mass- and stiffness matrices become, cf. Eqs. (28), (37), (38) and (39):

\[
\mathbf{M} = \frac{\mu l}{210} \begin{bmatrix} 78 & 11l \\ 11l & 2l^2 \end{bmatrix}
\]

(41)

\[
\mathbf{K}_e = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l \\ 6l & 4l^2 \end{bmatrix}
\]

(42)

\[
\mathbf{K}_g = \frac{\mu \Omega^2}{2940} \begin{bmatrix} 1260 & 189l \\ 189l & 70l^2 \end{bmatrix}
\]

(43)
The angular eigenfrequencies follows from the frequency condition, cf. Lecture 4, Eq. (43):

\[
\det \left( K_e + K_g - \omega^2 M \right) = 0
\]  

(44)

If the centrifugal stiffening effect is ignored (\( \Omega = 0 \)), the following angular eigenfrequencies are obtained:

\[
\omega_{n,0} = \begin{cases} 
3.5327 \sqrt{\frac{EI}{\mu l^4}} , & n = 1 \\
22.807 \sqrt{\frac{EI}{\mu l^4}} , & n = 2 
\end{cases}
\]  

(45)

Both are upper-bounds to the exact results as given by (31) as a result of the Rayleigh principle. The 1\textsuperscript{st} angular eigenfrequencies is acceptable, but the 2\textsuperscript{nd} is somewhat too big.
Introduction to the Finite Element Method

The finite element method is far the most general and applied numerical method for solving structural dynamic problems. The method is based on the division of the structure into a finite number of elements, each of which are required to be in dynamic equilibrium.

The so-called displacement (compatible) formulation of the method can be considered as a generalization of the displacement interpolation method described above.

Local dynamic equilibrium equations will only be derived for plane Bernoulli-Euler beam finite elements based on analytical dynamics. The geometric stiffness of the axial force will be ignored.
Equations of Motion of a Plane Bernoulli-Euler Beam Element

Figure 5: Plane Bernoulli-Euler beam element.
- \( q_1(t), \ldots, q_4(t) \): Nodal displacements and rotations. Degrees of freedom of the element.
- \( r_1(t), \ldots, r_4(t) \): Reaction forces and moments. These must balance the external loads, the inertial loads and the internal elastic and damping loads.
- \( EI, \mu \): Bending stiffness around the \( z \)-axis and mass per unit length. Both are constant along the beam element.
- \( l \): Length of beam element.
- \( u(x, t) \): Displacement field in the \( y \)-direction.
- \( p(x, t) \): Dynamic load per unit length in the \( y \)-direction.

The basic idea is to interpolate the displacement field, so this is compatible with the nodal displacements \( q_1(t), q_3(t) \) in the \( y \)-direction, and the nodal rotations \( q_2(t), q_4(t) \) in the \( z \)-direction:

\[
\begin{align*}
    u(0, t) &= q_1(t) ,
    & u(l, t) &= q_3(t) \\
    \frac{\partial u(0, t)}{\partial x} &= q_2(t) ,
    & \frac{\partial u(l, t)}{\partial x} &= q_4(t)
\end{align*}
\]

(46)
Cubic (Hermite) polynomial interpolation:

Figure 6: Shape functions of a plane beam element.
\[ u(x, t) = \Phi_1(x)q_1(t) + \Phi_2(x)q_2(t) + \Phi_3(x)q_3(t) + \Phi_4(x)q_4(t) \]  \hspace{1cm} (47)

where the shape functions \( \Phi_1(x), \ldots, \Phi_4(x) \) are given as:

\[
\begin{align*}
\Phi_1(x) &= 2\xi^3 - 3\xi^2 + 1 \\
\Phi_2(x) &= (\xi^3 - 2\xi^2 + \xi)l \\
\Phi_3(x) &= -2\xi^3 + 3\xi^2 \\
\Phi_4(x) &= (\xi^3 - \xi^2)l
\end{align*}
\hspace{1cm}, \quad \xi = \frac{x}{l} \hspace{1cm} (48)
\]

The shape function have been illustrated in Fig. 6.
(47) may be presented on the matrix form:

\[ u(x, t) = \Phi^T(x) q_e(t) = q_e^T(t) \Phi(x) \]  

(49)

\[ \Phi(x) = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \Phi_3(x) \\ \Phi_4(x) \end{bmatrix} \]  

(50)

\[ q_e(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix} \]  

(51)

- \( q_e(t) \) : Element degrees-of-freedom vector.
Curvature:

\[ \kappa(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2} = \]

\[
\frac{\partial^2}{\partial x^2} \left( \Phi_1(x)q_1(t) + \Phi_2(x)q_2(t) + \Phi_3(x)q_3(t) + \Phi_4(x)q_4(t) \right) = \\
\Phi_1''(x)q_1(t) + \Phi_2''(x)q_2(t) + \Phi_3''(x)q_3(t) + \Phi_4''(x)q_4(t) = \Phi''^T(x)q_e(t) \quad (52)
\]

\[
\Phi''(x) = \begin{bmatrix} 
\Phi_1''(x) \\
\Phi_2''(x) \\
\Phi_3''(x) \\
\Phi_4''(x)
\end{bmatrix} 
\quad (53)
\]

\[
\begin{aligned}
\Phi_1''(x) &= (12\xi - 6) \frac{1}{l^2} \\
\Phi_2''(x) &= (6\xi - 4) \frac{1}{l} \\
\Phi_3''(x) &= (-12\xi + 6) \frac{1}{l^2} \\
\Phi_4''(x) &= (6\xi - 2) \frac{1}{l}
\end{aligned}
\]

\[
\begin{aligned}
, \quad \xi &= \frac{x}{l}
\end{aligned}
\quad (54)\]
Kinetic and potential energy of a deformed beam element:

The kinetic energy expressed in the degrees of freedom of the beam element becomes:

\[
T(\dot{q}) = \int_0^l \frac{1}{2} \mu dx \left( \frac{\partial u}{\partial t} \right)^2 = \frac{1}{2} \int_0^l \mu \left( \Phi^T(x) \dot{q}_e(t) \right)^2 dx = \\
\frac{1}{2} \int_0^l \mu \dot{q}_e^T(t) \Phi(x) \Phi^T(x) \dot{q}(t) dx = \\
\frac{1}{2} \dot{q}_e^T(t) \left( \int_0^l \mu \Phi(x) \Phi^T(x) dx \right) \dot{q}_e(t) = \frac{1}{2} \dot{q}_e^T m_e \dot{q}_e(t) \tag{55}
\]

\[
m_e = \int_0^l \mu \Phi(x) \Phi^T(x) dx = \int_0^l \mu \begin{bmatrix}
\Phi_1^2 & \Phi_1 \Phi_2 & \Phi_1 \Phi_3 & \Phi_1 \Phi_4 \\
\Phi_2 \Phi_1 & \Phi_2^2 & \Phi_2 \Phi_3 & \Phi_2 \Phi_4 \\
\Phi_3 \Phi_1 & \Phi_3 \Phi_2 & \Phi_3^2 & \Phi_3 \Phi_4 \\
\Phi_4 \Phi_1 & \Phi_4 \Phi_2 & \Phi_4 \Phi_3 & \Phi_4^2
\end{bmatrix} dx = 
\]
The potential energy, i.e. the strain energy within the element plus the potential energy of the external load and the reaction forces and moments, becomes:

\[
U(q_e) = \int_0^l \frac{1}{2} EI \left( \frac{\partial^2 u(x, t)}{\partial x^2} \right)^2 dx - \int_0^l u(x, t)p(x, t) dx - r_1(t)u(0, t) - r_2(t)\frac{\partial u(0, t)}{\partial x} - r_3(t)u(l, t) - r_4(t)\frac{\partial u(l, t)}{\partial x}
\]

(57)
Insertion of (46), (49) and (52) provides:

\[
U(q_e) = \frac{1}{2} q_e^T(t) \left( \int_0^l EI \Phi''(x)\Phi''^T(x)dx \right) q_e(t) \\
- q_e^T(t) \int_0^l \Phi(x)p(x,t)dx - r_1(t)q_1(t) - r_2(t)q_2(t) - r_3(t)q_2(t) - r_4(t)q_4(t) = \\
\frac{1}{2} q_e^T(t)k_e q_e(t) - q_e^T(t)f_e(t) - q_e^T(t)r_e(t)
\]

(58)

\[
k_e = \int_0^l EI \Phi''(x)\Phi''^T(x)dx = \int_0^l EI \begin{bmatrix}
\Phi_1''&\Phi_2''&\Phi_3''&\Phi_4''\\
\Phi_1''&\Phi_2''&\Phi_3''&\Phi_4''\\
\Phi_3''&\Phi_2''&\Phi_3''&\Phi_4''\\
\Phi_4''&\Phi_4''&\Phi_3''&\Phi_4''
\end{bmatrix} dx = \\
\frac{EI}{l^3} \begin{bmatrix}
12 & 6l & -12 & 6l \\
6l & 4l^2 & -6l & 2l^2 \\
-12 & -6l & 12 & -6l \\
6l & 4l^2 & -6l & 4l^2
\end{bmatrix}
\]

(59)
\[ f_e(t) = \int_0^l \Phi(x)p(x,t)dx = \begin{bmatrix} \frac{1}{2}pl \\ \frac{1}{12}pl^2 \\ \frac{1}{2}pl \\ -\frac{1}{12}pl^2 \end{bmatrix} \] (60)

\[ r_e(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \\ r_4(t) \end{bmatrix} \] (61)

- \( k_e \): Element stiffness matrix.
- \( f_e(t) \): Element load vector.
- \( r_e(t) \): Element reaction vector.

The load vector (60) is only valid for a constant dynamic load per unit length, \( p(x,t) = p(t) \).
Lagrange’s equations of motion:

Lecture 4, Eqs. (10), (13):

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_e^T} \right) - \frac{\partial L}{\partial q_e^T} = f_{nc}(t) \quad \left( L(q_e, \dot{q}_e) = T(\dot{q}_e) - U(q_e) \right) \Rightarrow
\]

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_e^T} \right) + \frac{\partial U}{\partial q_e^T} = -c_e \dot{q}_e \quad \Rightarrow
\]

\[
m_e \ddot{q}_e + c_e \dot{q}_e + k_e q_e = \mathbf{f}_e(t) + \mathbf{r}_e(t)
\]

- \( c_e \): Element damping matrix. \( c_e = \alpha_0 m_e + \alpha_1 k_e \) (Rayleigh damping).
Example 3: Calculation of element load vector

The element load vector is calculated for the triangular load per unit length as shown in Fig. 7.

\[ p(x, t) = p_0(t) \frac{x}{l} \]

Figure 7: Triangular load per unit length.
\[ f_e(t) = \int_0^l \Phi(x)p_0(t)\frac{x}{l} \, dx = \frac{p_0(t)l}{60} \begin{bmatrix} 9 \\ 2l \\ 21 \\ -3l \end{bmatrix} \]  

(63)

It can be shown that the reaction forces and moment as given by Eq. (63) for \( q_e(t) = 0 \) are in static equilibrium with the specified triangular load, corresponding to \( f_e(t) + r_e(t) = 0 \).
Global Equations of Motion

- \((x, y, z)\) : Local coordinate system attached to the beam element.
- \((X, Y, Z)\) : Global coordinate system.
- \(q_e\) : Local components of degrees-of-freedom vector.
  Dimension: \(6 \times 1\).
- \(Q_e\) : Global components of degrees-of-freedom vector.
  Dimension: \(6 \times 1\).

Figure 8: Global and local coordinate systems.
$Q_e$ and $q_e$ are related as:

$$Q_e = A q_e$$  \hspace{1cm} (64)

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$  \hspace{1cm} (65)

$$a = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (66)

- $\psi$: Angle from global $X$-axis to local $x$-axis in the positive $Z$-direction (counter clock-wise).
The matrix \( a \) is *orthonormal* (the column vectors have the length 1 and are mutually orthogonal). Then:

\[
a^{-1} = a^T = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]  

(67)

Then, the inverse relation of (64) becomes:

\[
q_e = A^{-1} Q_e = A^T Q_e
\]  

(68)

The kinetic energy and potential energy of the element as given by (55) and (58) may be expressed in global components, using (68):

\[
T(\dot{q}_e) = \frac{1}{2} \dot{q}_e^T m_e \dot{q}_e = \frac{1}{2} \dot{Q}_e^T A m_e A^T Q_e = \frac{1}{2} \dot{Q}_e^T M_e \dot{Q}_e
\]  

(69)
The equations of motion of the element in global coordinates reads, cf. Eq. (62):

\[ U(\dot{q}_e) = \frac{1}{2} q_e^T k_e q_e - q_e^T (f_e(t) + r_e(t)) = \]

\[ \frac{1}{2} Q_e^T A k_e A^T Q_e - Q_e^T A \left( f_e(t) + r_e(t) \right) = \frac{1}{2} Q_e^T K_e Q_e - Q_e^T \left( F_e(t) + R_e(t) \right) \]

(70)

\[ M_e = A m_e A^T \] : Element mass matrix in global coordinates. \hspace{1cm} (71)

\[ K_e = A k_e A^T \] : Element stiffness matrix in global coordinates. \hspace{1cm} (72)

\[ F_e(t) = A f_e(t) \] : Element load vector in global coordinates. \hspace{1cm} (73)

\[ R_e(t) = A r_e(t) \] : Element reaction vector in global coordinates. \hspace{1cm} (74)
\[
M_e \ddot{Q}_e + C_e \dot{Q}_e + K_e Q_e = F_e(t) + R_e(t)
\]

- \(C_e\) : Element damping matrix in global coordinates. \(C_e = A c_e A^T\).

The kinetic and potential energy of the structure are made up by the sum of kinetic and potential energy of all \(m\) elements:

\[
T(\dot{Q}) = \sum_{j=1}^{m} T(\dot{Q}_{e,j}) = \sum_{j=1}^{m} \frac{1}{2} \dot{Q}_{e,j}^T M_{e,j} \dot{Q}_{e,j}
\]

\[
U(Q) = \sum_{j=1}^{m} U(Q_{e,j}) = \sum_{j=1}^{m} \left(\frac{1}{2} Q_{e,j}^T K_{e,j} Q_{e,j} - Q_{e,j}^T (F_{e,j} + R_{e,j})\right)
\]
Displacement and rotation components of adjacent elements at the interface are identical. The non-trivial degrees of freedom from the $n$ nodes of the structure may be assembled in the vector:

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix}$$  \hspace{1cm} (78)

- $n$: Number of nodes in the structure.

The dimension of $Q$ becomes $n = n_{dof} \times n$, where $n_{dof} = 3$ denotes the number of degrees of freedom per node. $Q_i$ stores the degrees of freedom related to node $i$. For a plane beam element this amounts to two displacement and one rotational degree of freedom. Since the structure is fixed at node 1, we have $Q_1 = 0$. 
At the interface between elements the components of the reaction forces \( R_{e,j} \) in the summation in (77) vanishes mutually. Only at the support nodes a net contributions occur, which are stored in vector \( \mathbf{R} \).

Then, (76) and (77) may be written as:

\[
T(\dot{\mathbf{Q}}) = \frac{1}{2} \dot{\mathbf{Q}}^T \mathbf{M} \dot{\mathbf{Q}}
\]

\[
U(\dot{\mathbf{Q}}) = \frac{1}{2} \dot{\mathbf{Q}}^T \mathbf{K} \mathbf{Q} - \mathbf{Q}^T (\mathbf{F} + \mathbf{R})
\]

- \( \mathbf{M} \): Global mass matrix. Dimension: \( n \times n \).
- \( \mathbf{K} \): Global stiffness matrix. Dimension: \( n \times n \).
- \( \mathbf{F} \): Global load vector. Dimension: \( n \times 1 \).
- \( \mathbf{R} \): Global reaction vector. Dimension: \( n \times 1 \).
Most components of $R$ are zero. Non-vanishing components only occur at the support nodes, and represent the reaction forces and reaction moments. The global equations of motion of the structure become:

$$M\ddot{Q} + C\dot{Q} + KQ = F(t) + R(t)$$  \hspace{1cm} (81)
Structure of $\mathbf{M}$ and $\mathbf{K}$:

\[
\mathbf{M} = \begin{bmatrix}
\mathbf{M}_{e,1} \\
\vdots \\
\mathbf{M}_{e,m-1} \\
\mathbf{M}_{e,m}
\end{bmatrix} \quad (82)
\]
Structure of $\mathbf{F}(t)$ and $\mathbf{R}(t)$ for a cantilever beam supported at node 1:

\[
\mathbf{F}(t) = \begin{bmatrix}
F_{e,1} \\
F_{e,2} \\
\vdots \\
F_{e,m-1} \\
F_{e,m}
\end{bmatrix}, \quad \mathbf{R}(t) = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
0
\end{bmatrix}
\] (83)
The subvector $\mathbf{R}_1(t)$ of dimension $n_{dof} \times 1$ determines the reaction forces and moment at the fixation. These are determined by the first $n_{dof}$ equations of (81) in combination with $Q_1 = 0$. The remaining equations determine the internal degrees of freedom.
Example 4: Cantilever beam modelled with two beam elements

Fig. 9 shows a plane Bernoulli-Euler cantilever beam with constant mass per unit length $\mu$ and constant bending stiffness $EI$ around the $y$-axis. The undamped eigenfrequencies are calculated by modelling the beam with two elements of the length $l/2$.

From (56), (59), (82), (83) follows:
\[ M = \frac{\mu l}{840} \begin{bmatrix} 156 & 11l & 54 & -\frac{13}{2}l & 0 & 0 \\ 11l & l^2 & \frac{13}{2}l & -\frac{3}{4}l^2 & 0 & 0 \\ 54 & \frac{13}{2}l & 156 & -11l & 0 & 0 \\ -\frac{13}{2}l & -\frac{3}{4}l^2 & -11l & l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 624 & 44l & 216 & -26l & 0 & 0 \\ 44l & 4l^2 & 26l & -3l & 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} \mu l \\ 840 \end{bmatrix} \begin{bmatrix} 0 & 0 & 156 & 11l & 54 & -\frac{13}{2}l \\ 0 & 0 & 11l & l^2 & \frac{13}{2}l & -\frac{3}{4}l^2 \\ 0 & 0 & 54 & \frac{13}{2}l & 156 & -11l \\ 0 & 0 & -\frac{13}{2}l & -\frac{3}{4}l^2 & -11l & l^2 \end{bmatrix} \]

\[ = \begin{bmatrix} M_{bb} \\ M_{bi} \\ M_{ib} \\ M_{ii} \end{bmatrix} \]

(84)
\[ \mathbf{K} = \frac{8EI}{l^3} \begin{bmatrix} 12 & 3l & -12 & 3l & 0 & 0 \\ 3l & l^2 & -3l & \frac{1}{2}l^2 & 0 & 0 \\ -12 & -3l & 12 & -3l & 0 & 0 \\ 3l & \frac{1}{2}l^2 & -3l & l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 8EI \hline l^3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 12 & 3l & -12 & 3l \\ 0 & 0 & 3l & l^2 & -3l & \frac{1}{2}l^2 \\ 0 & 0 & -12 & -3l & 12 & -3l \\ 0 & 0 & 3l & \frac{1}{2}l^2 & -3l & l^2 \\ 24 & 6l & -24 & 6l & 0 & 0 \\ 6l & 2l^2 & -6l & l^2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bi} \\ \mathbf{K}_{ib} & \mathbf{K}_{ii} \end{bmatrix} \] (85)
\[
\begin{align*}
Q &= \begin{bmatrix}
  Q_1 \\
  Q_2 \\
  Q_3 \\
  Q_4 \\
  Q_5 \\
  Q_6
\end{bmatrix} = \begin{bmatrix}
  Q_b \\
  Q_i
\end{bmatrix},
\quad 
R &= \begin{bmatrix}
  R_1 \\
  R_2 \\
  0 \\
  0 \\
  0
\end{bmatrix} = \begin{bmatrix}
  R_b \\
  0
\end{bmatrix}
\end{align*}
\]  
(86)

- **Index** \(b\): Boundary.
- **Index** \(i\): Interior.

Fixation at the hub \( \Rightarrow \)
\[
Q_b(t) \equiv 0
\]
(87)

Then, (81) may be rewritten as:
\[ R_b(t) = M_{bi} \ddot{Q}_i + K_{bi} Q_i \quad \text{(88)} \]
\[ M_{ii} \ddot{Q}_i + K_{ii} Q_i = 0 \quad \text{(89)} \]

The undamped angular frequencies follow from the characteristic equation, see Lecture 4, Eq. (43):

\[ \det(K_{ii} - \omega_i^2 M_{ii}) = 0 \quad \text{(90)} \]

The solutions of (89) become:

\[ \omega_n = \sqrt{\frac{EI}{\mu l^4}} \begin{cases} 
3.5177, & n = 1 \\
22.2215, & n = 2 \\
75.1571, & n = 3 \\
218.1380, & n = 4 
\end{cases} \quad \text{(91)} \]
The corresponding exact solutions become, see (31) and Lecture 9, Eq. (69):

\[
\omega_n = \sqrt{\frac{EI}{\rho l^4}} \cdot \begin{cases} 
3.5160 & , \ n = 1 \\
22.0345 & , \ n = 2 \\
61.6972 & , \ n = 3 \\
120.9032 & , \ n = 4 
\end{cases}
\] (91)

Only the lowest two eigenfrequencies are accurately calculated. In any case the numerical determined eigenfrequencies are upper bounds to the exact solutions, because compatible finite elements have been used (a consequence of the Rayleigh principle).

(MATLAB_2_Ex2)
Summary of Lecture 10

- Dynamical Modelling of Structures
  - Shape functions must fulfil the kinematic boundary conditions
    \[ \Phi_j(0) = \Phi'_j(0) = 0 \]. The mass matrix, stiffness matrix and the load vector are obtained by equating the kinetic energy and potential energy of the continuous and the discrete systems.
  - SDOF model. A single shape function.
  - 2DOF model. Two shape functions.