Outline of Lecture 6

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Multi-Degree-of-Freedom Systems (cont.)

Response to Harmonically Varying Loads

\[ M\ddot{x} + C\dot{x} + Kx = F(t) \quad , \quad t > 0 \]
\[ x(0) = x_0 \quad , \quad \dot{x}(0) = \dot{x}_0 \]

(1)

Figure 1: MDOF system. Harmonically varying dynamic loads and stationary responses.

Harmonically varying load vector \( F(t) \):
Let the dynamic load components be harmonically varying with the same angular frequency $\omega$ and different amplitudes $|F_{j,0}|$ and phases $\alpha_j$:

$$F(t) = \begin{bmatrix}
|F_{1,0}| \cos(\omega t - \alpha_1) \\
\vdots \\
|F_{n,0}| \cos(\omega t - \alpha_n)
\end{bmatrix} = \begin{bmatrix}
\text{Re}(F_{1,0}e^{i\omega t}) \\
\vdots \\
\text{Re}(F_{n,0}e^{i\omega t})
\end{bmatrix} = \text{Re}(F_0e^{i\omega t}) \quad (2)$$

$$F_0 = \begin{bmatrix}
F_{1,0} \\
\vdots \\
F_{n,0}
\end{bmatrix} = \begin{bmatrix}
|F_{1,0}|e^{-i\alpha_1} \\
\vdots \\
|F_{n,0}|e^{-i\alpha_n}
\end{bmatrix}: \text{Complex amplitude vector.} \quad (3)$$
Physical observation:

All components of the stationary motion (the particular solution) after dissipation of eigenvibrations caused by the initial conditions become harmonically varying with the same angular frequency $\omega$. The phases $\Psi_1, \ldots, \Psi_n$ of the response components are mutually different, and different from the phases $\alpha_1, \ldots, \alpha_n$ of the load components.

The stationary motion attains the form:

$$\mathbf{x}(t) = \begin{bmatrix} |X_1| \cos(\omega t - \Psi_1) \\ \vdots \\ |X_n| \cos(\omega t - \Psi_n) \end{bmatrix} = \begin{bmatrix} \text{Re}(X_1 e^{i\omega t}) \\ \vdots \\ \text{Re}(X_n e^{i\omega t}) \end{bmatrix} = \text{Re}(\mathbf{X}_0 e^{i\omega t}) \quad (4)$$

$$\mathbf{X}_0 = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} |X_1| e^{-i\Psi_1} \\ \vdots \\ |X_n| e^{-i\Psi_n} \end{bmatrix} \quad (5)$$
The complex amplitude vector $X_0$ contains information on all component amplitudes $|X_j|$ and phase angles $\Psi_j$.

Determination of $X_0$ by insertion of (2) and (4) into (1):

$$\ddot{x}(t) = \frac{d}{dt} \Re(X_0 e^{i\omega t}) = \Re(i\omega X_0 e^{i\omega t})$$

$$\dddot{x}(t) = \frac{d^2}{dt^2} \Re(X_0 e^{i\omega t}) = \Re((i\omega)^2 X_0 e^{i\omega t})$$

$$M\ddot{x} + C\dot{x} + Kx = \Re\left(\left((i\omega)^2 M + i\omega C + K\right)X_0 e^{i\omega t}\right) = \Re\left(F_0 e^{i\omega t}\right) \Rightarrow$$

$$\Re\left(\left(-\omega^2 M + i\omega C + K\right)X_0 - F_0\right) e^{i\omega t} = 0 \Rightarrow$$
\[ (-\omega^2 M + i\omega C + K) \mathbf{X}_0 - \mathbf{F}_0 = 0 \quad \Rightarrow \]

\[ \mathbf{X}_0 = \mathbf{H}(\omega)\mathbf{F}_0 \quad (6) \]

\[ \mathbf{H}(\omega) = (\mathbf{K} - \omega^2 \mathbf{M} + i\omega \mathbf{C})^{-1} \quad \text{: Frequency response matrix.} \quad (7) \]

The system matrices in modal and physical coordinates are connected as follows:

\[
\begin{align*}
\mathbf{m} & = \mathbf{S}^T \mathbf{M} \mathbf{S} \\
\mathbf{k} & = \mathbf{S}^T \mathbf{K} \mathbf{S} \\
\mathbf{c} & = \mathbf{S}^T \mathbf{C} \mathbf{S} \\
\mathbf{M} & = (\mathbf{S}^{-1})^T \mathbf{m} \mathbf{S}^{-1} \\
\mathbf{K} & = (\mathbf{S}^{-1})^T \mathbf{k} \mathbf{S}^{-1} \\
\mathbf{C} & = (\mathbf{S}^{-1})^T \mathbf{c} \mathbf{S}^{-1}
\end{align*}
\]

\[ \left( (\mathbf{S}^T)^{-1} = (\mathbf{S}^{-1})^T \right) \quad (9) \]

where the modal matrix is given by \( \mathbf{S} = \begin{bmatrix} \Phi^{(1)} & \Phi^{(2)} & \cdots & \Phi^{(n)} \end{bmatrix} \), and its inverse is evaluated from \( \mathbf{S}^{-1} = \mathbf{m}^{-1} \mathbf{S}^T \mathbf{M} \), cf. Lecture 5, Eq. (75).
Then,

\[ H(\omega) = \left( (S^{-1})^T (k - \omega^2 m + i\omega c) S^{-1} \right)^{-1} \]

\[ = S(k - \omega^2 m + i\omega c)^{-1} S^T = Sh(\omega)S^T \]  

(10)

\[ h(\omega) = (k - \omega^2 m + i\omega c)^{-1} = \begin{bmatrix} H_1(\omega) \\ & \ddots \\ & & H_n(\omega) \end{bmatrix} \]  

(11)

\[ H_j(\omega) = \frac{1}{k_j - \omega^2 m_j + i\omega c_j} = \frac{1}{m_j(\omega_j^2 - \omega^2 + 2\zeta_j\omega_j\omega i)} \]  

(12)

- \( H_j(\omega) \): SDOF frequency response function of the modal equation of motion: \( m_j \ddot{q}_j + c_j \dot{q}_j + k_j q_j = f_j(t) \), cf. Lecture 5, Eq. (70).
\[ S = \begin{bmatrix} \Phi^{(1)} & \Phi^{(2)} & \ldots & \Phi^{(n)} \end{bmatrix} \Rightarrow \]

\[ H(\omega) = (K - \omega^2 M + i\omega C)^{-1} = Sh(\omega)S^T = \sum_{j=1}^{n} H_j(\omega)\Phi^{(j)}\Phi^{(j)T} \quad (13) \]

Especially for \( \omega = 0 \):

\[ K^{-1} = \sum_{j=1}^{n} \frac{1}{k_j} \Phi^{(j)}\Phi^{(j)T} \quad (14) \]

\[ \left( H(0) = K^{-1} , \quad H_j(0) = \frac{1}{k_j} = \frac{1}{m_j\omega_j^2} \right) \]

(14) represents an expansion of the inverse stiffness matrix (the flexibility matrix) in terms of outer products of the eigenmodes. (14) is known as Mercer’s theorem.
Damping Models

In modal analysis the explicit form of the damping matrix is not needed. We simply enter the modal damping ratios $\zeta_j$ in the modal equations of motion solve them one by one:

$$m_j (\ddot{q}_j + 2\zeta_j \omega_j \dot{q}_j + \omega_j^2 q_j) = f_j(t), \quad j = 1, \ldots, n_1$$

$$\mathbf{x}(t) \approx \sum_{j=1}^{n_1} \Phi^{(j)} q_j(t)$$

(15)

In Eq. (15) the modal expansion has been truncated after $n_1$ terms, where $n_1$ denotes the number of rigid and elastic modes of importance for the response. Clearly, the method requires that the eigensolutions $(\omega_j, \Phi^{(j)}), \quad j = 1, \ldots, n_1$ are available.
Numerical analysis of the dynamic response focus directly on the matrix differential equation Eq. (1). The idea is to omit the costly and tedious initial determination of the eigensolutions (ωj, Φ(j)). Instead the damping matrix C must be estimated in a way, so it displays the known or prescribed damping properties of the structure. Further, this calibration should be established based on a minimum of needed eigensolutions.
Rayleigh’s Damping Model

Typically, only a few damping ratios, say $\zeta_1$ and $\zeta_2$ for the lowest two modes, are known. Then, a damping matrix representing these damping ratios can be constructed by the so-called Rayleigh damping model, also known as proportional damping. In this case the damping matrix is obtained as a linear combination of $M$ and $K$:

$$ C = \alpha_0 M + \alpha_1 K $$

(16) implies that the modal decoupling condition Lecture 5, Eq. (67) is fulfilled. The relation between the corresponding modal matrices becomes, cf. Eq. (8):

$$ c = \alpha_0 m + \alpha_1 k $$

(17)
The relation between the diagonal components in the involved matrices become:

\[ c_j = \alpha_0 m_j + \alpha_1 k_j \quad \Rightarrow \]

\[ 2 \zeta_j \omega_j m_j = \alpha_0 m_j + \alpha_1 \omega_j^2 m_j \quad \Rightarrow \]

\[ \zeta_j = \frac{1}{2\omega_j} \alpha_0 + \frac{\omega_j}{2} \alpha_1, \quad j = 1, \ldots, n \]  

(18)

For \( j = 1, 2 \), Eq. (18) provides the following relation for calibration of \( \alpha_0 \) and \( \alpha_1 \):

\[
\begin{bmatrix}
\zeta_1 \\
\zeta_2
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2\omega_1} & \frac{\omega_1}{2} \\
\frac{1}{2\omega_2} & \frac{\omega_2}{2}
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\
\alpha_1
\end{bmatrix} \Rightarrow
\]

\[
\begin{bmatrix}
\alpha_0 \\
\alpha_1
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2\omega_1} & \frac{\omega_1}{2} \\
\frac{1}{2\omega_2} & \frac{\omega_2}{2}
\end{bmatrix}^{-1}
\begin{bmatrix}
\zeta_1 \\
\zeta_2
\end{bmatrix}
\]

(19)
With $\alpha_0, \alpha_1$ determined by (19), Eq. (16) will represent the 1st and 2nd modal damping ratios correctly. Higher modal damping ratios are determined by (18), and may be different from the actual (although unknown) modal damping ratios.

Obviously, the calibration of the Rayleigh damping model requires that the undamped angular frequencies $\omega_1$ and $\omega_2$ are available. No knowledge of the corresponding eigenmodes $\Phi^{(1)}$ and $\Phi^{(2)}$ is needed.
Example 1: Alternative calibration of Rayleigh’s damping model

Figure 2: Calibration of Rayleigh damping to minimum damping ratio $\zeta_j = \zeta_{\text{min}} = 0.01$ at the angular eigenfrequency $\omega_j = \omega_{\text{min}} = 12\,s^{-1}$. 
Alternatively, the Rayleigh damping model may be calibrated to provide a prescribed damping ratio $\zeta_j = \zeta_{\text{min}}$ at a certain angular frequency $\omega_j = \omega_{\text{min}}$ in a way that the damping ratio at all other modes are higher. The minimum condition follows from (18):

$$
\zeta_{\text{min}} = \frac{\alpha_0}{2\omega_{\text{min}}} + \frac{\alpha_1}{2} \omega_{\text{min}} \quad \Rightarrow
$$

$$
\frac{d\zeta_{\text{min}}}{d\omega} = -\frac{\alpha_0}{2\omega_{\text{min}}^2} + \frac{\alpha_1}{2} = 0 \quad \Rightarrow
$$

$$
\alpha_0 = \alpha_1 \omega_{\text{min}}^2 \quad \Rightarrow
$$

$$
\zeta_{\text{min}} = \alpha_1 \omega_{\text{min}} \quad \Rightarrow
$$

$$
\begin{align*}
\alpha_0 &= \zeta_{\text{min}} \omega_{\text{min}} \\
\alpha_1 &= \zeta_{\text{min}} \omega_{\text{min}} \omega_{\text{min}}^{-1} \\
\end{align*}
$$

(20)

The indicated calibration suggested by Krenk guaranties a certain minimum damping in all modes. High and low frequency modes are related with high damping.
Cauchey's damping model is a generalization of Rayleigh's damping model. In this case a number $n_1 > 2$ of damping ratios $\zeta_1, \zeta_2, \ldots, \zeta_{n_1}$ are available. The basis of the model is matrix products of the type:

$$K_l = \left( KM^{-1} \right)^l M = \begin{cases} 
\underbrace{KM^{-1} \cdots KM^{-1}}_{l \text{ factors}} M, & l = 1, 2, \ldots \\
M, & l = 0 \\
\underbrace{(KM^{-1})^{-1} \cdots (KM^{-1})^{-1}}_{-l \text{ factors}} M, & l = -1, -2, \ldots 
\end{cases}$$

Notice that $(KM^{-1})^{-1} = MK^{-1}.$
$K_l$ fulfills the following orthogonality property:

$$S^T K_l S = (k m^{-1})^l m, \quad l = 0, \pm 1, \pm 2, \cdots$$  \hspace{1cm} (22)

where $m$ and $k$ are the modal mass and stiffness matrices.

**Proof:**

The modal mass and stiffness matrices are given as, cf. Eq. (8):

\[
\begin{align*}
    m & = S^T M S \\
    k & = S^T K S \\
    M & = (S^T)^{-1} m S^{-1} \\
    K & = (S^T)^{-1} k S^{-1} \\
    M^{-1} & = S m^{-1} S^T \\
    K^{-1} & = S k^{-1} S^T
\end{align*}
\]

\hspace{1cm} (23)
Then,
\[ KM^{-1} = (S^T)^{-1} km^{-1} S^T \]
\[ (KM^{-1})^{-1} = (S^T)^{-1} (km^{-1})^{-1} S^T \] \[ \quad \Rightarrow \]
\[ (KM^{-1})^l = (S^T)^{-1} (km^{-1})^l S^T \quad , \quad l = 0, \pm 1, \pm 2, \ldots \quad (24) \]

Finally, from (23) and (24):
\[ (KM^{-1})^l M = (S^T)^{-1} (km^{-1})^l m S^{-1} \quad , \quad l = 0, \pm 1, \pm 2, \ldots \quad (25) \]

(22) follows immediately from (25).

The right hand side of (22) is a diagonal matrix. Then, the left hand side is also diagonal, and hence must be symmetric:
\[ S^T K_l S = (S^T K_l S)^T = S^T K_l^T S \]
\[ \Rightarrow \]
\[ K_l = K_l^T \quad \text{(Symmetric for arbitrary } l \text{)} \quad (26) \]
The Caughey damping matrix is obtained as a linear combination of arbitrary but different matrix products of the type $K_l$:

$$C = \sum_{p=1}^{n_1} a_{l_p} K_{l_p}$$  \hspace{1cm} (27)

where the indices $l_1, \ldots, l_{n_1}$ are arbitrary selected integers. Especially, it follows from (26) that the Caughey damping matrix is symmetric. The Rayleigh model is obtained for $n_1 = 2$, and for the indices $l_1 = 0 (K_0 = M)$ and $l_2 = 1 (K_1 = (KM^{-1})M = K)$. 
The modal mass and modal stiffness are related as\[ \frac{k_j}{m_j} = \omega_j^2 = \lambda_j, \] cf. Lecture 5, Eq. (54). Then, the diagonal matrix \((k \ m^{-1})_l \ m\) has the structure:

\[
(k \ m^{-1})_l \ m = S^T K_l S = \begin{bmatrix}
\omega_1^{2l} m_1 \\
\vdots \\
\omega_n^{2l} m_n
\end{bmatrix}
\]

where \(l = 0, \pm 1, \pm 2, \ldots\)
From (27) and (28) follows that the components in the diagonal of the modal damping matrix $\mathbf{c}$ are determined from:

$$c_j = 2\zeta_j \omega_j m_j = \sum_{p=1}^{n_1} a_{lp} \omega_j^{2lp} m_j \quad \Rightarrow$$

$$\zeta_j = \frac{1}{2} \sum_{p=1}^{n_1} a_{lp} \omega_j^{2lp-1}, \quad j = 1, 2, \ldots, n \quad (29)$$

If $\zeta_j$ and $\omega_j$ are known for $j = 1, \ldots, n_1$ the coefficients $a_{lp}$ in (27) can be calculated from (29) for arbitrary selections of the indices $l_1, \ldots, l_{n_1}$.

The Caughey model only requires knowledge of $(\zeta_j, \omega_j), \quad j = 1, \ldots, n_1$ but not of the related eigenmodes $\Phi(j)$.

The Caughey model will represent the damping ratios of the first $n_1$ modes correctly. The damping ratios of the higher modes are determined from (29), and may be different from the actual (unknown) damping ratios.
Example 2: Caughey damping models for $n_1 = 3$

Assume that $n_1 = 3$, so that $\zeta_1, \zeta_2, \zeta_3$ are known. The indices $l_p$ are chosen as $l_1 = 0$, $l_2 = 1$ and $l_3 = 2$, corresponding to the following damping model:

$$C = a_0 M + a_1 K + a_2 K M^{-1} K$$  \hspace{1cm} (30)

where it has been used that $K_2 = (K M^{-1}) (K M^{-1}) M = K M^{-1} K$. $a_0, a_1, a_2$ are determined from (29) as:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{\omega_1} & \omega_1 & \omega_3^3 \\ \frac{1}{\omega_2} & \omega_2 & \omega_2^3 \\ \frac{1}{\omega_3} & \omega_3 & \omega_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$  \hspace{1cm} (31)
Alternatively, one may choose \( l_1 = -1, \ l_2 = 0 \) and \( l_3 = 1 \), which leads to the damping model:

\[
C = a_{-1} MK^{-1} M + a_0 M + a_1 K
\]  

(32)

where it has been used that \( K^{-1} = (K M^{-1})^{-1} M = MK^{-1} M \). Next, \( a_{-1}, a_0, a_1 \) are determined from:

\[
\begin{bmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\frac{1}{\omega_1^3} & \frac{1}{\omega_1} & \omega_1 \\
\frac{1}{\omega_2^3} & \frac{1}{\omega_2} & \omega_2 \\
\frac{1}{\omega_3^3} & \frac{1}{\omega_3} & \omega_3
\end{bmatrix} \begin{bmatrix}
a_{-1} \\
a_0 \\
a_1
\end{bmatrix}
\]  

(33)

If \( n = 3 \) it can be shown that the damping matrices (30) and (32) becomes completely identical. If \( n > 3 \) the matrices are different, but they both represent the modal damping ratios \( \zeta_1, \zeta_2, \zeta_3 \) of the lowest 3 modes correctly.
State Vector Formulation of Equations of Motion

Linear SDOF system:

\[
\begin{align*}
    m\ddot{x} + c\dot{x} + kx &= f(t), \quad t > t_0 \\
    x(t_0) &= x_0, \quad \dot{x}(t_0) = \dot{x}_0
\end{align*}
\]  

(34) is equivalent to the following system of two 1st order equations:

\[
\begin{align*}
    \frac{d}{dt}x &= \dot{x} \quad \text{(dummy equation)} \\
    \frac{d}{dt}\dot{x} &= -\frac{k}{m} - \frac{c}{m}\dot{x} + \frac{1}{m}f(t)
\end{align*}
\]  

⇒
\[
\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} f(t) \end{bmatrix} \quad (35)
\]

(35) may be written in the vector format:

\[
\frac{d}{dt} \mathbf{z}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{f}(t) \quad , \quad t > t_0 \\
\mathbf{z}(t_0) = \mathbf{z}_0
\]

\[
\mathbf{z}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} , \quad \mathbf{z}_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} , \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ \frac{1}{m} f(t) \end{bmatrix} \quad (36)
\]

\[
\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}
\]

- \( \mathbf{z}(t) \) : State vector of the dynamic system.

A state vector formulation of the equations of motion is presumed in many numerical algorithms.
Linear MDOF system:

\[
\begin{align*}
M\ddot{x} + C\dot{x} + Kx &= F(t), \quad t > t_0 \\
x(t_0) &= x_0, \quad \dot{x}(t_0) = \dot{x}_0
\end{align*}
\]

Equivalent state vector formulation:

\[
\begin{align*}
\frac{d}{dt} x &= \dot{x} \quad \text{(dummy vector equation)} \\
\frac{d}{dt} \dot{x} &= -M^{-1}Kx - M^{-1}C\dot{x} + M^{-1}F(t)
\end{align*}
\]
\[
\frac{d}{dt} z(t) = Az(t) + f(t), \quad t > t_0 \\
z(t_0) = z_0
\]

\[
A = \begin{bmatrix}
0 & I \\
-M^{-1}K & -M^{-1}C
\end{bmatrix}
\]

(36), (39) represent the generic state vector formulation for linear dynamic systems.
Nonlinear MDOF system:

\[ M\ddot{x} + g(x, \dot{x}) = F(t) , \quad t > 0 \]
\[ x(0) = x_0 , \quad \dot{x}(0) = \dot{x}_0 \]

\[ \frac{d}{dt} z(t) = h(z, t) , \quad t > 0 \]
\[ z(0) = z_0 \]

\[ z(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} , \quad z_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} , \quad h(z, t) = \begin{bmatrix} \dot{x}(t) \\ -M^{-1}g(x, \dot{x}) + M^{-1}F(t) \end{bmatrix} \]

(41)

(42)
Example 3: State vector formulation of the equations of motion for a mathematical double pendulum

Figure 3: Mathematical double pendulum.
Equations of motion, cf. Lecture 4, Eqs. (57), (58):

\[
\begin{align*}
\ddot{\theta}_1 + \omega_1^2 \theta_1 - \frac{1}{2} (\omega_2^2 - \omega_1^2) (\theta_2 - \theta_1) &= 0 \\
\ddot{\theta}_2 + \omega_1^2 \theta_2 + \frac{1}{2} (\omega_2^2 - \omega_1^2) (\theta_2 - \theta_1) &= 0
\end{align*}
\]  

(43)

where

\[
\begin{align*}
\omega_1^2 &= \frac{g}{l} \\
\omega_2^2 &= \frac{g}{l} + 2 \frac{k}{m} \frac{a^2}{l^2}
\end{align*}
\]  

(44)
State-vector formulation of (43):
\[
\dot{z} = h(z), \quad t > t_0 \\
z(t_0) = z_0
\]

where
\[
z(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{bmatrix}, \quad h(z) = \begin{bmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \\ -\omega_1^2 \theta_1 + \frac{1}{2} (\omega_2^2 - \omega_1^2)(\theta_2 - \theta_1) \\ -\omega_1^2 \theta_2 - \frac{1}{2} (\omega_2^2 - \omega_1^2)(\theta_2 - \theta_1) \end{bmatrix}
\]

\[
z_0 = \begin{bmatrix} \theta_1(t_0) \\ \theta_2(t_0) \\ \dot{\theta}_1(t_0) \\ \dot{\theta}_2(t_0) \end{bmatrix}
\]

(45)
Numerical time integration implies the determination of the motion of the structure as described by the state vector $z^T(t) = [x(t), \dot{x}(t)]$ at discrete instants of time $t_n = t_0 + n\Delta t$, $n = 1, 2, \ldots$ separated by the time step $\Delta t$, given that the initial value $z_0$ at the time $t_0$ is known, and that the external dynamic load vector $F(t)$ of the matrix equation of (1) can be calculated at the indicated instants of time.

The principle of numerical time integration has been summarized in Box 1.
Box 1: Principle of numerical time-integration of structural dynamic systems

1. Perform a discretization of the time axis (i.e. select a time step $\Delta t$). The initial value of the state vector $z^T_0 = [x_0, \dot{x}_0]$ at the time $t = t_0$ is known.

2. For $n = 0, 1, \ldots$ calculate the new state vector $z^T_{n+1} = [x_{n+1}, \dot{x}_{n+1}]$, based on the state vector $z_n = z(t_n)$, $z^T_n = [x_n, \dot{x}_n]$ and the load vector $F_n = F(t_n)$ of the matrix differential equation at the previous time $t_n = t_0 + n\Delta t$.

Errors in numerical time integration:

- **Truncation error**: Deviation between exact and numerical determination of $z_n$ in a single time step, given that $z_{n-1}$ is known.
- **Global error**: Deviation between exact and numerical determination of $z_n$ in $n$ time steps, given that $z_0$ is known.
**$k^{th}$ order method:**

It can be shown that if the truncation error is of the magnitude $O(\Delta t^{k+1})$, then the global error is of the magnitude $O(\Delta t^k)$.

For a $k^{th}$ order method the global error is of the magnitude $O(\Delta t^k)$.

**Stability:**

Lack of stability means that the numerical scheme explodes exponentially with time. Stability requires that the fraction $\frac{\Delta t}{T_{\text{min}}}$ is smaller than a certain critical value, where $T_{\text{min}}$ is the period of the highest mode in the structural system.

**Unconditional stability:**

The numerical algorithm produces a finite result (although not necessary an accurate result) for arbitrary large time step $\Delta t$. 
Euler Scheme

Taylor expansion of $z(t_n + \Delta t)$:

$$z(t_n + \Delta t) = z(t_n) + \dot{z}(t_n) \Delta t + \frac{1}{2!} \ddot{z}(t_n) \Delta t^2 + \cdots + \frac{1}{n!} z^{(n)}(t_n + \theta \Delta t) \Delta t^n$$

(47)

- $\frac{1}{n!} z^{(n)}(t_n + \theta \Delta t) \Delta t^n = \frac{1}{n!} \frac{d^n}{dt^n} z(t + \theta \Delta t) \Delta t^n$: Lagrange’s remainder.
- $\theta \in ]0, 1[$: Different for each component in $z(t)$.

(47) is truncated after the first two terms on the right hand side. Further, $\ddot{z}(t_n) = h(z_n, t_n)$ is inserted, where $h(z_n, t_n)$ is the right hand side of (39) or (41) calculated at the previous solution at the time $t_n$. Then:
\[ z(t_n + \Delta t) = z(t_n) + h(z_n, t_n)\Delta t + O(\Delta t^2) \Rightarrow \]
\[ z_{n+1} \approx z_n + h_n \Delta t, \quad h_n = h(z_n, t_n) \]  

(48)

Figure 5: Euler integration, one-dimensional case.
The method has been illustrated in Fig. 5 for a one-dimensional case. At the time $t_n$ the slope $h_n = h(z_n, t_n)$ is calculated based on the approximate solution $z_n$. The solution is linear in the interval $[t_n, t_{n+1}]$ with the indicated slope. The Euler algorithm has been indicated in Box 2.

**Box 2 : Euler scheme**

1. Select the time step $\Delta t$. The initial state vector $z_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}$ is known.

2. For $n = 0, 1, \cdots$ calculate:

$$h_n = h(z_n, t_n)$$  \hspace{1cm} (49)

$$z_{n+1} = z_n + h_n \Delta t$$  \hspace{1cm} (50)
As seen from Fig. 5 the exact and the numerical solutions deviate increasingly as \( n \to \infty \). This is the case for all numerical schemes. Because the truncation error is of the magnitude \( O(\Delta t^2) \), the Euler scheme is a 1\(^{st}\) order method (low accuracy scheme). Further, the method is not especially stable as demonstrated in the following Example 4.
Example 4: Integration of the equations of motion for a mathematical double pendulum by means of an Euler scheme

The initial values are given as:

\[
\begin{align*}
\theta_1(0) &= \theta_{1,0}, & \dot{\theta}_1(0) &= 0 \\
\theta_2(0) &= 0, & \dot{\theta}_2(0) &= 0
\end{align*}
\]

The exact solution can be shown to be, cf. Lecture 4, Eq. (59) and Lecture 5, Eq. (25):

\[
\begin{bmatrix}
\theta_1(t) \\
\theta_2(t)
\end{bmatrix} = a_1 \Phi^{(1)}(t) \cos(\omega_1 t) + a_2 \Phi^{(2)}(t) \cos(\omega_2 t) + b_1 \Phi^{(1)}(t) \sin(\omega_1 t) + b_2 \Phi^{(2)}(t) \sin(\omega_2 t) + \frac{1}{2} \theta_{1,0} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t) \right)
\]

\[(52)\]
Further, it is assumed that:

\[ \begin{align*}
\omega_1 &= 1 \text{ s}^{-1} \\
\omega_2 &= 10 \text{ s}^{-1}
\end{align*} \]  

(53)

The equations of motion (45), (46) are integrated by means of an Euler scheme using the time steps:

\[ \Delta t = 0.01T_2 \quad , \quad \Delta t = 0.0002T_2 \quad , \quad \Delta t = 0.0001T_2 \]  

(54)

where \( T_1 = \frac{2\pi}{\omega_2} \) is the eigenperiod of the highest modes. As seen from Figs. 6 and 7 the numerical algorithm becomes unstable for \( \Delta t = 0.0002T_2 \).

MATLAB file: Euler_Scheme.m
Figure 6: Euler scheme. Rotational angle of 1st pendulum of a mathematically double pendulum, $T_1 = 10T_2$. Black curve: Analytical solution. a) $\Delta t = 0.01T_2$. b) $\Delta t = 0.0002T_2$. c) $\Delta t = 0.0001T_2$. 
Figure 7: Euler scheme. Difference of rotational angles of a mathematically double pendulum, $T_1 = 0.1T_0$. Black curve: Analytical solution. a) $\Delta t = 0.01T_2$. b) $\Delta t = 0.0002T_2$. c) $\Delta t = 0.0001T_2$. 
4th Order Runge-Kutta Scheme

The 4th order Runge-Kutta scheme has a truncation error equal to $O(\Delta t^5)$. Hence, the method is a 4th order method.

Still, the stability depends on $\frac{\Delta t}{T_{\min}}$. For large dimensional system, $T_{\min}$ can be very small (i.e. $\omega_{\max} = \frac{2\pi}{T_{\min}}$ is very large). Hence, the time step is controlled by numerical stability rather than by accuracy.

The Runge-Kutta algorithm has been indicated in Box 3.
Box 3: $4^{th}$ order Runge-Kutta scheme

1. Select the time step $\Delta t$. The initial state vector $z_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix}$ is known.

2. For $n = 0, 1, \cdots$ calculate:

$$h_1 = h(z_n, t_n)$$

$$h_2 = h \left( z_n + \frac{1}{2} \Delta th_1, t_n + \frac{1}{2} \Delta t \right)$$

$$h_3 = h \left( z_n + \frac{1}{2} \Delta th_2, t_n + \frac{1}{2} \Delta t \right)$$

$$h_4 = h(z_n + \Delta th_3, t_n + \Delta t)$$

$$z_{n+1} = z_n + \frac{\Delta t}{6} (h_1 + 2h_2 + 2h_3 + h_4)$$
Example 5: Integration of the equations of motion for a mathematical double pendulum by means of a 4\textsuperscript{th} order Runge-Kutta scheme

Perform the same calculations as in Exercise 4 by means of a 4\textsuperscript{th} order Runge-Kutta scheme with the time steps:

\[ \Delta t = 0.45T_2 \quad , \quad \Delta t = 0.452T_2 \quad , \quad \Delta t = 0.1T_2 \]  \hspace{1cm} (57)

As seen from Figs. 8 and 9 the numerical algorithm becomes unstable for \( \Delta t \simeq 0.45T_2 \).

MATLAB file: Runge_Kutta_Scheme.m
Figure 8: 4th order Runge-Kutta scheme. Rotational angle of 1st pendulum of a mathematically double pendulum, $T_1 = 10T_2$. Black curve: Analytical solution. a) $\Delta t = 0.451T_2$. b) $\Delta t = 0.450T_2$. c) $\Delta t = 0.11T_2$. 
Figure 9: 4th order Runge-Kutta scheme. Difference of rotational angles of a mathematically double pendulum, $T_1 = 10T_2$. Black curve: Analytical solution.  

a) $\Delta t = 0.451T_2$.  
b) $\Delta t = 0.450T_2$.  
c) $\Delta t = 0.1T_2$. 

- Lowest half of eigenmodes: Carries the dynamic response.
- Highest half of eigenmodes: Are not accurately determined by the GEVP. These should be considered merely as spatial discretization noise.

**Euler and 4th Runge-Kutta schemes:**
Conditional stable integration schemes. The time step is determined from the highest mode in the structural system due to stability requirements. The Euler and the 4th order Runge-kutta schemes are examples of so-called explicit time integration algorithms, i.e. no solution of linear or nonlinear equations for the state vector is needed at the new time step.

**Reasonable criterion for selection of the time step:**
Accurate determination of the lowest modes, which determines the dynamic response. This motivates the interest for unconditional stable time integration algorithms in structural dynamics (no instability induced by high frequency modes. The determination of these modes is inaccurate).
Summary of Lecture 6

- Response to Harmonically Varying Loads.

$$F(t) = \begin{bmatrix} |F_{1,0}| \cos(\omega t - \alpha_1) \\ \vdots \\ |F_{n,0}| \cos(\omega t - \alpha_n) \end{bmatrix} = \begin{bmatrix} \text{Re}(F_{1,0}e^{i\omega t}) \\ \vdots \\ \text{Re}(F_{n,0}e^{i\omega t}) \end{bmatrix} = \text{Re}(F_0e^{i\omega t})$$

$$F_{j,0} = |F_{j,0}|e^{-i\alpha_j}$$

$$x(t) = \begin{bmatrix} |X_1| \cos(\omega t - \Psi_1) \\ \vdots \\ |X_n| \cos(\omega t - \Psi_n) \end{bmatrix} = \begin{bmatrix} \text{Re}(X_1e^{i\omega t}) \\ \vdots \\ \text{Re}(X_ne^{i\omega t}) \end{bmatrix} = \text{Re}(X_0e^{i\omega t})$$

$$X_j = |X_j|e^{-i\Psi_j}$$
\[ X_0 = H(\omega)F_0 \]

\[ H(\omega) = (K - \omega^2 M + i\omega C)^{-1} : \text{Frequency response matrix.} \]

\[ H(\omega) = \sum_{j=1}^{n} H_j(\omega)\Phi(j)\Phi(j)^T \]

\[ H_j(\omega) = \frac{1}{m_j(\omega_j^2 - \omega^2 + 2\zeta_j \omega_j \omega \omega i)} : \text{Modal frequency response function.} \]

- **Damping Models**
  - Primarily to be used in numerical time integration. Only a few angular eigenfrequencies are needed.
    - Rayleigh’s Damping Model.
    - Caughey’s Damping Model.
State Vector Formulations of Equations of Motion.

\[ z(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \]

Numerical Time Integration.

- Euler Scheme.
- 4th Order Runge-Kutta Scheme.

Explicit time integration schemes are conditional stable.