Outline of Lecture 8

- Multi-Degree-of-Freedom Systems (cont.)
  - Symmetric Structures.
  - Vibrations due to Indirectly Acting Dynamic Loads.
  - Tuned Mass Dampers.
    - Formulation of 2DOF Model.
    - Analysis and Design of Tuned Mass Dampers.
Symmetric Structures

Often a structure possess some symmetry planes, which can be used to reduce the number of degrees of freedom of the system. System reduction based on symmetry is not related with loss of accuracy.

System reduction requires the existence of one or more symmetry planes in the structure. In structural dynamics symmetry is required with respect to:

- Geometry.
- Boundary conditions.
- Stiffness distribution.
- Damping distribution.
- Mass distribution.

The dynamic loads and the initial conditions are not assumed to fulfill any symmetry or anti-symmetry conditions.
Figure 1: Plane symmetric structure. a) Partitioning of dynamic loads into anti-symmetric and symmetric components. b) Equivalent system for analysis of anti-symmetric vibrations. c) Equivalent system for analysis of symmetric vibrations.
Fig. 1 shows a plane structure with 10 degrees of freedom, which fulfill the indicated symmetry conditions. The dynamic loads may be partitioned into symmetric and anti-symmetric load distributions as shown on Fig. 1a.

The anti-symmetric load distribution causes zero vertical displacement and a point of inflection at the cross-sections originally placed in the symmetric plane. Hence, the anti-symmetric vibrations may be analyzed by the system with 5 degrees of freedom shown in Fig. 1b. Notice that the mass $\frac{1}{2}m_3$ is related to the horizontal degree of freedom $y_5 = x_5$, and that the vertical load $F_2(t)$ placed in the symmetry plane can be totally ignored.

The symmetric load distribution causes zero horizontal displacement and zero tangential slopes at the cross-sections in the symmetry plane. Then, symmetric vibrations may be analyzed by the system with 5 degrees of freedom shown in Fig. 1c.
The mass $\frac{1}{2}m_3$ is now related to the vertical degree of freedom $x_6$, and the horizontal load $F_3(t)$ acting in the symmetry plane is ignored.

It follows from the example that if a system has a symmetry plane, the system of $n$ degrees of freedom can be reduced to two equivalent systems of $\frac{n}{2}$ degrees of freedom.

The calculation time of a dynamic system typically increases proportional to $n^3$. Then, the saving in computational effort for structures with a single symmetry plane is reduced by a factor $2 \left( \frac{1}{2} \right)^3 = \frac{1}{4}$. 
Example 1: Reduction of a symmetric 2DOF system to 2 SDOF systems.

\[ x_1(0) = x_{1,0}, \quad x_2(0) = x_{2,0} \]
\[ \dot{x}_1(0) = \dot{x}_{1,0}, \quad \dot{x}_2(0) = \dot{x}_{2,0} \]

\[ y_1(0) = \frac{1}{2}(x_1(0) + x_2(0)) \]
\[ \dot{y}_1(0) = \frac{1}{2}(\dot{x}_1(0) + \dot{x}_2(0)) \]

\[ y_2(0) = \frac{1}{2}(x_1(0) - x_2(0)) \]
\[ \dot{y}_2(0) = \frac{1}{2}(\dot{x}_1(0) - \dot{x}_2(0)) \]

Figure 2: Symmetric 2DOF system. Partitioning of dynamic loads and initial conditions into anti-symmetric and symmetric components.
The 2DOF system shown in Fig. 2 is dynamically symmetric about the midpoint. The dynamic loads and inertial conditions may be partitioned into anti-symmetric and symmetric components as shown.

![Diagram of 2DOF system]

\[ y_1(0) = \frac{1}{2} (x_{1,0} + x_{2,0}) \]
\[ \dot{y}_1(0) = \frac{1}{2} (\dot{x}_{1,0} + \dot{x}_{2,0}) \]

Figure 3: Equivalent SDOF system for the analysis of anti-symmetric vibrations.

The anti-symmetric load and anti-symmetric initial conditions cause the masses to move in the same direction with the same displacement \( y_1(t) \) as shown on Fig. 2. Then, the spring and damper between the masses are not deformed, and may be disregarded. Hence, \( y_1(t) \) can be analyzed by the SDOF system shown on Fig. 3, corresponding to the equation of motion:
The symmetric loads and initial conditions cause the masses to move in opposite directions with the same displacement \( y_2(t) \) as shown on Fig. 2.

\[
m\ddot{y}_1 + c\dot{y}_1 + ky_1 = \frac{1}{2}(f_1(t) + f_2(t)) \quad , \quad t > 0
\]

\[
y_1(0) = \frac{1}{2}(x_{1,0} + x_{2,0}) \quad , \quad \dot{y}_1(0) = \frac{1}{2}(\dot{x}_{1,0} + \dot{x}_{2,0})
\]

Figure 4: Equivalent spring and dampers during symmetric vibrations.

During symmetric vibration, points at the symmetric line must be at rest. This can be achieved by replacing the spring with the constant \( 2k \) and the damper with the constant \( 2c \) with 2 equivalent springs and dampers in series with the constants \( K \) and \( C \) as shown on Fig. 4.
The spring force on the masses in the original system during symmetric motions is equal to \(2k(2y_2) = 4ky_2\). The spring forces on the masses for the system shown on Fig. 4 is equal to \(Ky_2\). These must be equal, leading to:

\[Ky_2 = 4ky_2 \quad \Rightarrow \quad K = 4k\]  

(2)

Similarly, the damping force on the masses in the original system during symmetric motions is equal to \(2c(2\dot{y}_2) = 4c\dot{y}_2\). The damper force on the masses for the system shown on Fig. 4 is equal \(C\dot{y}_2\). These must be equal, leading to:

\[C\dot{y}_2 = 4c\dot{y}_2 \quad \Rightarrow \quad C = 4c\]  

(3)
Then, $y_2(t)$ can be analyzed by the SDOF system shown on Fig. 5, corresponding to the equation of motion:

$$m \ddot{y}_2 + 5c \dot{y}_2 + 5ky_2 = \frac{1}{2} (f_1(t) - f_2(t)) \quad , \quad t > 0$$

$$y_2(0) = \frac{1}{2} (x_{1,0} + x_{2,0}) \quad , \quad \dot{y}_2(0) = \frac{1}{2} (\dot{x}_{1,0} + \dot{x}_{2,0})$$

The original degrees of freedom are recovered from the solution to $y_1(t)$ and $y_2(t)$, obtained from (1) and (4):

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) + y_2(t) \\ y_1(t) - y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$
(5) unveals $y_1(t)$ and $y_2(t)$ as modal coordinates, cf. Lecture 5, Eq. (46). Actually, the eigenmodes of the system are:

$$\Phi^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \Phi^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Vibrations due to Indirectly Acting Dynamic Loads

Figure 6: MDOF system exposed to an indirectly acting dynamic load.
An MDOF system is exposed to an external dynamic force $f(t)$ acting indirectly on the masses via the linear elastic structure. The motion of the masses $m_1, \ldots, m_n$ are described by the degrees of freedom $x_1, \ldots, x_n$ measured from the static equilibrium state, see Fig. 6.

Additionally, an auxiliary degree of freedom $x_0$ is introduced co-directional to the dynamic force $f(t)$.

The deformations $x_i$, $i = 0, 1, \ldots, n$ are brought forward as a sum of contributions from the external force $f(t)$ acting at $x_0$ and from the inertial forces $f_{Ij}(t)$ and the damping forces $f_{dj}(t)$ acting in the positive and negative direction of the degrees of freedom $x_j$. Then, by the use of the force method the following identities may be formulated:
\begin{align*}
x_0(t) &= \delta_{00} f(t) + \delta_{01} (f_{I1} - f_{d1}) + \cdots + \delta_{0n} (f_{In} - f_{dn}) \\
x_1(t) &= \delta_{10} f(t) + \delta_{11} (f_{I1} - f_{d1}) + \cdots + \delta_{1n} (f_{In} - f_{dn}) \\
\vdots \\
x_n(t) &= \delta_{n0} f(t) + \delta_{n1} (f_{I1} - f_{d1}) + \cdots + \delta_{nn} (f_{In} - f_{dn})
\end{align*}

where:

\begin{align*}
f_{di} &= \sum_{j=1}^{n} C_{ij} \ddot{x}_j \\
f_{Ii} &= -\sum_{j=1}^{n} M_{ij} \ddot{x}_j
\end{align*}
(7), (8) and (9) may be written on the following matrix form:

\[ x_0(t) = \delta_{00} f(t) - d_0^T (M\ddot{x} + C\dot{x}) \]  
\[ x(t) = d_0 f(t) - D (M\ddot{x} + C\dot{x}) \]

where:

\[ d_0 = \begin{bmatrix} \delta_{10} \\ \vdots \\ \delta_{n0} \end{bmatrix}, \quad D = \begin{bmatrix} \delta_{11} & \cdots & \delta_{1n} \\ \vdots & \ddots & \vdots \\ \delta_{n1} & \cdots & \delta_{nn} \end{bmatrix} \]

\[ x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \]

Upon pre-multiplication with \( K = D^{-1} \), (11) attains the form:

\[ M\ddot{x} + C\dot{x} + Kx = Kd_0 f(t) \]
Example 2: Indirectly acting dynamic load on a 2DOF system.

Figure 7: Plane Bernoulli-Euler frame exposed to a horizontal, indirectly acting dynamic force.
Fig. 7 shows a plane framed structure made up of massless Bernoulli-Euler beams with the bending stiffness $EI$. At the free end is attached a point mass $m$ and a vertical acting linear viscous damper with the damper constant $c$. The structure is exposed to a horizontal dynamic load $f(t)$ at the midst of the vertical beam.

The system has 2 degrees of freedom $x_1(t)$ and $x_2(t)$ defined as the horizontal and vertical displacement of the point mass as shown in Fig. 7. Additionally, a horizontal auxiliary degree $x_0(t)$ at the dynamic force is introduced.

The flexibility coefficient $\delta_{ij}$ and $\delta_{i0}$ are calculated by the use of the principle of complementary virtual work, cf. Lecture 3, Eq. (54):

$$D = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} = \frac{1}{3} \frac{a^3}{EI} \begin{bmatrix} 8 & -6 \\ -6 & 7 \end{bmatrix}$$ (15)

$$d_0 = \begin{bmatrix} \delta_{10} \\ \delta_{20} \end{bmatrix} = \frac{1}{6} \frac{a^3}{EI} \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$ (16)
The stiffness matrix becomes:

\[ K = D^{-1} = \frac{3}{20} \frac{EI}{a^3} \begin{bmatrix} 7 & 6 \\ 6 & 8 \end{bmatrix} \]

The equations of motion become, cf. (14):

\[ M\ddot{x} + C\dot{x} + Kx = Kd_0 f(t) \quad (17) \]

\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad Kd_0 = \frac{1}{40} \begin{bmatrix} 17 \\ 6 \end{bmatrix} \quad (18) \]

\[ M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \]
A tuned mass damper (TMD) is a linear SDOF oscillator with a mass $m_d$, a spring with the constant $k_d$ and a viscous damper with the damper constant $c_d$, attached to a given MDOF structure. The TMD is referred to as the secondary system, and the MDOF system as the primary system.

Figure 8: MDOF system (primary system) with attached tuned mass damper (secondary system).
The idea is to transfer mechanical energy from the primary system to the secondary system, so the primary system only perform moderate vibrations. The secondary system is allowed to undergo substantial vibrations, which are dissipated by the damper $c_d$.

The *tuning* of the mass damper refers to the selection of the parameters $k_d$, $c_d$ and $m_d$ in a way that maximum mechanical energy is transferred from the primary system to the secondary system for a given dynamic loading. Here, we shall presume that the external dynamic loads on the primary structure are harmonically varying, cf. Lecture 6, Eq. (2):

$$F(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix} = \begin{bmatrix} |F_1| \cos(\omega t - \alpha_1) \\ \vdots \\ |F_n| \cos(\omega t - \alpha_n) \end{bmatrix} = \begin{bmatrix} \text{Re}(F_1 e^{i\omega t}) \\ \vdots \\ \text{Re}(F_n e^{i\omega t}) \end{bmatrix} \tag{19}$$

$$F_j = |F_j| e^{-i\alpha_j} \tag{20}$$
Formulation of 2DOF model

Figure 9: a) Eigenmode of damped mode. b) Definition of degrees of freedom of 2DOF model.
A TMD can only damp a single mode of the structural response effectively.

If several modes are dominating the response a corresponding number of mass dampers must be attached, each tuned to a separate mode.

Fig. 9 shows a TMD acting in the direction of the $j$th degree of freedom $x_j$. The TMD is supposed to damp the mode with the eigenmode $\Phi$ shown on Fig. 9a. Although the fundamental mode has been depicted, $\Phi$ may be any eigenmode. The only requirement is that the $j$th component of $\Phi$ fulfills:

$$\Phi_j \neq 0 \quad (21)$$

(21) implies that the TMD is able to perform a work on the considered mode. In this case the mode is said to be controllable.
The modal decomposition of the displacement vector $\mathbf{x}(t)$ is assumed to be dominated by the mode to be damped:

$$\mathbf{x}(t) = \sum_{k=1}^{n} \Phi^{(k)} q_k(t) \simeq \Phi q(t) \quad (22)$$

where $q(t)$ is the modal coordinate of the critical mode. (22) implies that the displacement of the primary structure at the point of attachment of the TMD becomes:

$$x(t) = x_j(t) \simeq \Phi_j q(t) \quad (23)$$

The TMD is cut free from the structure, and the internal force $u(t)$ is applied as an external force on the structure and the TMD, with the signs shown on Fig. 9b. $u(t)$ is denoted the control force.
Totally, the primary structure is loaded with the external dynamic load vector \( F^T(t) = [f_1(t), \ldots, f_n(t)] \) and the control force \( u(t) \) acting in the direction of the \( j \)th degree of freedom \( x_j(t) \). Then, the equation of motion for the modal coordinate \( q(t) \) becomes, cf. Lecture 5, Eqs. (69), (70):

\[
m \ddot{q} + c \dot{q} + kq = f(t) + \Phi_j u(t)
\]

- \( m \) : Modal mass.  \hspace{1cm} m = \Phi^T M \Phi.
- \( c \) : Modal damping.  \hspace{1cm} c = \Phi^T C \Phi.
- \( k \) : Modal stiffness.  \hspace{1cm} k = \Phi^T K \Phi.
- \( f(t) \) : Modal load of external loads.  \hspace{1cm} f(t) = \Phi^T F(t).
Notice that the modal load from the control force becomes \( \Phi_j u(t) \). With the dynamic load vector given by (19) the modal load from the external loads becomes:

\[
f(t) = \Phi_1 \text{Re} \left( F_1 e^{i\omega t} \right) + \cdots + \Phi_n \text{Re} \left( F_n e^{i\omega t} \right)
\]

\[
= \text{Re} \left( \left( \Phi_1 |F_1| e^{-i\alpha_1} + \cdots + \Phi_n |F_n| e^{-i\alpha_n} \right) e^{i\omega t} \right)
\]

\[
= \text{Re} \left( f_0 e^{-i\omega t} \right)
\]

where complex amplitude \( f_0 \) of the modal load is given as:

\[
f_0 = \Phi_1 |F_1| e^{-i\alpha_1} + \cdots + \Phi_n |F_n| e^{-i\alpha_n}
\]
Use of (23) in (24) provides the following equation of motion formulated in the displacement $x(t)$ of the point of attachment of the TMD:

$$M \ddot{x} + C \dot{x} + K x = F(t) + u(t)$$  \hspace{1cm} (27)

$$M = \frac{m}{\Phi_j^2} = \frac{\Phi^T M \Phi}{\Phi_j^2} , \quad C = \frac{c}{\Phi_j^2} = \frac{\Phi^T C \Phi}{\Phi_j^2}$$

$$K = \frac{k}{\Phi_j^2} = \frac{\Phi^T K \Phi}{\Phi_j^2} , \quad F(t) = \frac{f(t)}{\Phi_j} = \text{Re} \left( F_0 e^{i\omega t} \right)$$  \hspace{1cm} (28)

$M$, $C$, $K$ and $F(t)$ are physical quantities (i.e. independent of the normalization of the eigenmode $\Phi$). These are only equal to the modal mass, the modal damping and the modal load, if $\Phi$ is normalized so the $j$th component $\Phi_j = 1$. The complex force amplitude caused by the external dynamic load becomes $F_0 = \frac{f_0}{\Phi_j}$, where $f_0$ is given by (26).
The equation of the free TMD with the control force acting in the opposite direction of the displacement degree of freedom \( x_d \) of the damper mass becomes, see Fig. 9b:

\[
m_d \ddot{x}_d + c_d \dot{x}_d + k_d x_d = -u(t)
\]  

(29)

Figure 10: Two-degree-of-freedom model of primary and secondary system.

The internal control force is related to selected degrees of freedom as:

\[
u(t) = k_d (x_d - x) + c_d (\dot{x}_d - \dot{x})
\]  

(30)
Elimination of $u(t)$ in (27) and (29) by means of (30) provides the following equations of motion of the system:

$$
\begin{align*}
M\ddot{x} + C\dot{x} + Kx - k_d(x_d - x) - c_d(x_d - x) &= F(t) \\
m_d\ddot{x}_d + c_d\dot{x}_d + k_d\dot{x}_d + k_d(x_d - x) + c_d(x_d - x) &= 0
\end{align*}
$$

(31)

(31) may be recasted into the following matrix equation of motion:

$$
M\ddot{\boldsymbol{x}} + C\dot{\boldsymbol{x}} + K\boldsymbol{x} = F(t) = \text{Re} \left( F_0 e^{i\omega t} \right)
$$

(32)

$$
\boldsymbol{\varphi}(t) = \begin{bmatrix} \varphi(t) \\ \varphi_d(t) \end{bmatrix} , \quad \boldsymbol{F}_0 = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \\
\boldsymbol{M} = \begin{bmatrix} M & 0 \\ 0 & m_d \end{bmatrix} , \quad \boldsymbol{C} = \begin{bmatrix} C + c_d & -c_d \\ -c_d & c_d \end{bmatrix} , \quad \boldsymbol{K} = \begin{bmatrix} K + k_d & -k_d \\ -k_d & k_d \end{bmatrix}
$$

(33)
The TMD will only damp the primary structure, if \( u(t) \) performs a negative net work on the primary structure during the period \( T = \frac{2\pi}{\omega} \), see Fig. 9b:

\[
\int_0^T u(t) \dot{x}(t) dt < 0
\]  

(34)

The control force \( u(t) \) may very well perform a positive net work on other modes than the controlled one, so these modes are excited rather than damped. This phenomena is denoted control spillover.

Normally, the modal damping of the primary structure is very low (else there are no need of the TMD). Hence, this damping will be ignored in the following analysis:

\[
C' = 0
\]  

(35)
Analysis and Design of Tuned Mass Dampers

The stationary harmonic response of the system (32) is given by, cf. Lecture 6, Eqs. (4-7):

\[
x(t) = \text{Re} \left( \begin{bmatrix} X e^{i\omega t} \\ X_d e^{i\omega t} \end{bmatrix} \right)
\]

where the complex amplitudes \( X \) and \( X_d \) of \( x(t) \) and \( x_d(t) \) are determined from the linear equations:

\[
\begin{bmatrix}
K + k_d + i\omega c_d - \omega^2 M & -k_d - i\omega c_d \\
-k_d - i\omega c_d & k_d + i\omega c_d - \omega^2 m_d
\end{bmatrix}
\begin{bmatrix}
X \\
X_d
\end{bmatrix}
= \begin{bmatrix}
F_0 \\
0
\end{bmatrix}
\]

(37)
$F_0$ is the complex amplitude of the excitation defined next to Eq. (28). From (37) the following solutions are obtained for $X$ and $X_d$:

$$
\begin{align*}
\frac{X}{F_0} & = \frac{k_d + i\omega c_d - \omega^2 m_d}{D} \\
\frac{X_d}{F_0} & = \frac{k_d + i\omega c_d}{D}
\end{align*}
$$

where $D$ is the determinant of the coefficient matrix in (37), given as:

$$D = (K + k_d + i\omega c_d - \omega^2 M)(k_d + i\omega c_d - \omega^2 m_d) - (k_d + i\omega c_d)^2$$

The following parameters are introduced:

$$
\begin{align*}
\omega_0^2 & = \frac{K}{M} , \quad \mu = \frac{m_d}{M} \\
\omega_d^2 & = \frac{k_d}{m_d} , \quad \zeta_d = \frac{c_d}{2\sqrt{k_d m_d}}
\end{align*}
$$
\( \omega_0 \) indicates the undamped angular eigenfrequency of the considered mode of the primary structure. \( \mu \) represents the relevant mass ratio. \( \omega_d \) and \( \zeta_d \) may be interpreted as the angular eigenfrequency and damping ratio of the secondary system, when the primary system is fixed \((X = 0)\). Then, the solutions (38) for the complex amplitudes may be written in the following non-dimensional forms:

\[
\begin{align*}
\frac{X}{F_0/K} &= \frac{\omega_0^2 (\omega_d^2 - \omega^2 + 2\zeta_d \omega_d \omega i)}{(\omega_0^2 - \omega^2) (\omega_d^2 - \omega^2) - \mu \omega_d^2 \omega^2 + 2\zeta_d \omega_d \omega (\omega_0^2 - (1 + \mu) \omega^2) i} \\
\frac{X_d}{F_0/K} &= \frac{\omega_0^2 (\omega_d^2 + 2\zeta_d \omega_d \omega i)}{(\omega_0^2 - \omega^2) (\omega_d^2 - \omega^2) - \mu \omega_d^2 \omega^2 + 2\zeta_d \omega_d \omega (\omega_0^2 - (1 + \mu) \omega^2) i}
\end{align*}
\]

(41)

Notice that \(|F_0/K|\) denotes the quasi-static displacement amplitude of the primary structure for \( \omega \to 0 \). Hence, \(|X/F_0/K|\) denotes the dynamic amplification of the primary structure, cf. Lecture 2, Eq. (42).
Figure 11: Dynamic amplification factor of the primary system, $\mu = 0.05, \omega_d = \omega_0$.

- - - : $\zeta_d = 0.1$.
- - - : $\zeta_d = 0$.
- - - : $\zeta_d = \infty$. 
Fig. 11 shows the dynamic amplification factor of the primary system for the tuning $\mu = 0.05$, $\omega_d = \omega_0$ and for various values of $\zeta_d$. All curves attain the value $\left| \frac{X}{F_0/K} \right| = 1$ for $\omega = 0$ due to the normalization with respect to the quasi-static response.

For $\zeta_d = 0 \Leftrightarrow c_d = 0$ infinite resonance peaks exist at the angular frequencies $\omega_1$ and $\omega_2$, which represent the undamped angular eigenfrequencies of the 2DOF system (32). The dynamic amplification factor follows from (41) for $\zeta_d = 0$:

$$\left| \frac{X}{F_0/K} \right| = \left| \frac{\omega_0^2 (\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2) (\omega_d^2 - \omega^2) - \mu \omega_d^2 \omega^2} \right|$$  \hspace{1cm} (42)

$\omega_1$ and $\omega_2$ are determined as the values of $\omega$ for which the denominator of (42) vanishes, i.e. as solution to the 2nd order equation:

$$(\omega_0^2 - \omega^2) (\omega_d^2 - \omega^2) - \mu \omega_d^2 \omega^2 = 0$$ \hspace{1cm} (43)
For $\zeta_d = \infty$ the damper is locking the relative motion between the primary and the secondary system, and the combined system acts as an undamped SDOF system with the mass $M + m_d$ and the stiffness $K$. The amplification factor follows from (41) for $\zeta_d \to \infty$:

$$\left| \frac{X}{F_0/K} \right| = \left| \frac{\omega_0^2}{\omega_0^2 - (1 + \mu) \omega^2} \right|$$

(44)

The angular resonance frequency $\omega_\infty$ becomes:

$$\omega_\infty = \sqrt{\frac{K}{M + m_d}} = \omega_0 \sqrt{\frac{1}{1 + \mu}}$$

(45)

For $\zeta_d \neq 0$, 2 resonance peaks appears at angular frequencies slightly above $\omega_1$ and below $\omega_2$, as shown on Fig. 11.
As first demonstrated by Den Hartog the family of amplification curves (41) obtained for varying damping ratio $\zeta_d$ possesses the remarkable property that they all passes through two points $P$ and $Q$ placed at the angular excitation frequencies $\omega_P$ and $\omega_Q$, where:

$$\omega_1 < \omega_P < \omega_0 < \omega_Q < \omega_2$$  \hspace{1cm} (46)

The angular frequencies $\omega_P$ and $\omega_Q$ are essential at the optimal tuning of the damper. Next, these are determined.

(41a) is written on the form:

$$\frac{X}{F_0/K} = \frac{a + 2\zeta_db_i}{c + 2\zeta_ddi}$$  \hspace{1cm} (47)
where:

\[ a = \omega_0^2 (\omega_d^2 - \omega^2) \]
\[ b = \omega_0^2 \omega_d \omega \]
\[ c = (\omega_0^2 - \omega^2) (\omega_d^2 - \omega^2) - \mu \omega_d^2 \omega^2 \]
\[ d = \omega_d \omega (\omega_0^2 - (1 + \mu) \omega^2) \]

\begin{equation}
\left\{ \begin{array}{l}
 a = \omega_0^2 (\omega_d^2 - \omega^2) \\
 b = \omega_0^2 \omega_d \omega \\
 c = (\omega_0^2 - \omega^2) (\omega_d^2 - \omega^2) - \mu \omega_d^2 \omega^2 \\
 d = \omega_d \omega (\omega_0^2 - (1 + \mu) \omega^2)
\end{array}\right. \tag{48}
\end{equation}

From (47) follows

\[ \left| \frac{X}{F_0/K} \right|^2 = \frac{a^2 + 4\zeta_d^2 b^2}{c^2 + 4\zeta_d^2 d^2} = \frac{a^2}{c^2} \left( \frac{1 + 4\zeta_d^2 \frac{b^2}{a^2}}{1 + 4\zeta_d^2 \frac{d^2}{c^2}} \right) \]

\begin{equation}
\left| \frac{X}{F_0/K} \right|^2 = \frac{a^2 + 4\zeta_d^2 b^2}{c^2 + 4\zeta_d^2 d^2} = \frac{a^2}{c^2} \left( \frac{1 + 4\zeta_d^2 \frac{b^2}{a^2}}{1 + 4\zeta_d^2 \frac{d^2}{c^2}} \right) \tag{49}
\end{equation}
(49) is independent of $\zeta_d$, if:

\[
\frac{b^2}{a^2} = \frac{d^2}{c^2} \quad \Rightarrow \quad \frac{b}{a} = \pm \frac{d}{c} \quad \Rightarrow
\]

\[
\frac{\omega_0^2 \omega_d \omega}{\omega_0^2 (\omega_d^2 - \omega^2)} = \pm \frac{\omega_d \omega (\omega_0^2 - (1 + \mu) \omega^2)}{(\omega_0^2 - \omega^2) (\omega_d^2 - \omega^2) - \mu \omega_d^2 \omega^2} \quad \Rightarrow
\]

\[
\frac{1}{\omega_d^2 - \omega^2} = \pm \frac{\omega_0^2 - (1 + \mu) \omega^2}{(\omega_0^2 - \omega^2) (\omega_d^2 - \omega^2) - \mu \omega_d^2 \omega^2}
\]

(50)

The plus sign leads to the unique solution $\omega^2 = 0$, which corresponds to the quasi-static response. The solutions searched for are obtained for the minus sign:

\[
\frac{1}{\omega_d^2 - \omega^2} = -\frac{\omega_0^2 - (1 + \mu) \omega^2}{(\omega_0^2 - \omega^2) (\omega_d^2 - \omega^2) - \mu \omega_d^2 \omega^2} \quad \Rightarrow
\]

\[
(2 + \mu) \omega^4 - 2 (\omega_0^2 + (1 + \mu) \omega_d^2) \omega^2 + 2 \omega_0^2 \omega_d^2 = 0
\]

(51)
\( \omega_P^2 \) and \( \omega_Q^2 \) are the two solutions of (51). The sum of these becomes:

\[
\omega_P^2 + \omega_Q^2 = \frac{2}{2 + \mu} \left( \omega_0^2 + (1 + \mu) \omega_d^2 \right)
\]  

(52)

The amplifications factors at the points \( P \) and \( Q \) are independent of \( \zeta_d \), and hence can be calculated from (44) valid for \( \zeta_d = \infty \):

\[
\left| \frac{X}{F_0/K} \right|_{P,Q} = \pm \frac{\omega_0^2}{\omega_0^2 - (1 + \mu) \omega_{P,Q}^2}
\]  

(53)

where the + sign refers to point \( P \) and the − sign to point \( Q \). The amplification factor at these points are required to be identical:

\[
\frac{\omega_0^2}{\omega_0^2 - (1 + \mu) \omega_P^2} = - \frac{\omega_0^2}{\omega_0^2 - (1 + \mu) \omega_Q^2}
\]  

⇒

\[
\omega_P^2 + \omega_Q^2 = \frac{2}{1 + \mu} \omega_0^2
\]  

(54)
The tuning angular frequency \( \omega_d^2 \) follows from the identity of the right hand sides of (52) and (54):

\[
\frac{2}{2 + \mu} \left( \omega_0^2 + (1 + \mu) \omega_d^2 \right) = \frac{2}{1 + \mu} \omega_0^2 \quad \Rightarrow 
\]

\[
\omega_d = \frac{1}{1 + \mu} \omega_0 
\]

(55)
Figure 12: Dynamic amplification factors of the primary system for $\mu = 0.05$, $\omega_d = \frac{1}{1+\mu} \omega_0$.

- $\zeta_d = 0.1$.
- $\zeta_d = \sqrt{\frac{1}{2} \frac{\mu}{1+\mu}}$.
- $\zeta_d = \infty$. 
We still need one more fixation of the dynamic amplification curve in order to specify the damping ratio $\zeta_d$. Krenk suggested a simple criteria for this by requiring that the amplification factor at the angular frequency $\omega_\infty$ should be equal to the amplification factor at the angular frequencies $\omega_P$ and $\omega_Q$.

At first, the amplification factor at $\omega_P$ and $\omega_Q$ is determined. For the tuning (55), Eq. (51) attains the form:

\[(2 + \mu) \omega^4 - 2 \left(1 + \frac{1}{1 + \mu}\right) \omega_0^2 \omega^2 + 2 \left(\frac{1}{1 + \mu}\right)^2 \omega_0^4 = 0 \quad \Rightarrow \]

\[\omega^4 - 2\omega_\infty^2 \omega^2 + \frac{2}{2 + \mu} \omega_\infty^4 = 0 \quad (56)\]

where (45) has been used in the last statement. (56) has the solution:

\[\omega_{P,Q} = \omega_\infty^2 \left(1 \pm \sqrt{\frac{\mu}{2 + \mu}}\right) \quad (57)\]
(57) is inserted into (44) to provide:

\[
\left| \frac{X}{F_0/K} \right|_{P,Q} = \frac{\omega_0^2}{\omega_0^2 - (1 + \mu) \omega^2 \left( 1 - \sqrt{\frac{\mu}{2 + \mu}} \right)} = \sqrt{\frac{2 + \mu}{\mu}} \quad (58)
\]

Next, the amplification factor is calculated at the angular frequency \( \omega_\infty \) by means of (41). After insertion of \( \omega = \omega_\infty \) given by (45) and \( \omega_d \) given by (55), the following result is obtained after some reductions:

\[
\frac{X(\omega_\infty)}{F_0/K} = \frac{\mu - 2\zeta_d \sqrt{1 + \mu} \; i}{\mu} \quad \Rightarrow \\
\left| \frac{X(\omega_\infty)}{F_0/K} \right| = \sqrt{\frac{\mu^2 + 4\zeta_d^2 (1 + \mu)}{\mu^2}} \quad (59)
\]
Finally, equating (58) and (59) provides the following solution for $\zeta_d$:

$$\sqrt{\frac{2 + \mu}{\mu}} = \sqrt{\frac{\mu^2 + 4\zeta_d^2(1 + \mu)}{\mu^2}} \quad \Rightarrow$$

$$\zeta_d = \sqrt{\frac{\mu}{2(1 + \mu)}}$$

(60)

We are interested in an equivalent modal damping ratio $\zeta$ of the primary structure as a measure of the effect of the TMD. At resonance $\omega = \omega_0$ the amplification factor is $\frac{1}{2\zeta}$, cf. Lecture 2, Eq. (24). Additionally, with the considered calibration the amplification factor is flat in the interval $[\omega_P, \omega_Q]$ as shown on Fig. 12, so the value at $\omega = \omega_0$ is approximately given by (58). Using $\mu \ll 1$, it follows:

$$\frac{1}{2\zeta} \approx \sqrt{\frac{2 + \mu}{\mu}} \approx \sqrt{\frac{2}{\mu}} \quad \Rightarrow \quad 2\zeta \approx \sqrt{\frac{\mu}{2}}$$

(61)
Similarly, $\zeta_d$ given by (60) may be approximated as:

$$\zeta_d = \sqrt{\frac{\mu}{2(1 + \mu)}} \approx \sqrt{\frac{\mu}{2}} \approx 2\zeta \quad \Rightarrow$$

$$\zeta \approx \frac{1}{2} \zeta_d$$

(62)

This concludes the analysis.

The design procedure goes in the reverse order. At first a demanded damping ratio $\zeta$ of the primary structure is prescribed. Then, the necessary damping ratio $\zeta_d$ of the secondary structure is determined from (62). Next, the mass ratio $\mu$ is determined from (60). Then, $\omega_d$ can be calculated from (55). Finally, $m_d$, $c_d$, $k_d$ can be determined from (40). The various steps in the design procedure have been summarized in Box 1.
Box 1: Design procedure of tuned mass damper

An MDOF linear structure defined by the mass matrix $M$ and the stiffness matrix $K$ shall be damped in a specific mode with the mode shape $\Phi$ and the angular eigenfrequency $\omega_0$ by means of a tuned mass damper with the mass $m_d$, the damper constant $c_d$ and the spring stiffness $k_d$. The damper is acting co-directional to the $j$th degree of freedom with the mode shape component $\Phi_j$.

At the design of the tuned mass damper, i.e. the determination of $m_d$, $c_d$ and $k_d$, the following steps are followed:

1. Determine the mass and stiffness parameters $M$ and $K$ of the primary system by Eq. (28):

$$M = \frac{\Phi^T M \Phi}{\Phi_j^2}, \quad K = \omega_0^2 M$$  \hspace{1cm} (63)
2. Specify the required modal damping $\zeta$ of the considered mode of the primary system.

3. Calculate the damping ratio of the secondary system $\zeta_d$ from Eq. (62):

$$\zeta_d = 2\zeta$$  \hspace{1cm} (64)

4. Calculate the mass ratio $\mu$ from Eq. (60):

$$\mu = \frac{2\zeta_d^2}{1 - 2\zeta_d^2}$$  \hspace{1cm} (65)

5. Calculate the angular frequency of the secondary system $\omega_d$ from Eq. (55):

$$\omega_d = \frac{1}{1 + \mu} \omega_0$$  \hspace{1cm} (66)
6. Calculate $m_d$, $c_d$ and $k_d$ from Eq. (40):

\begin{align*}
    m_d & = \mu M \quad \text{(67)} \\
    k_d & = \omega_d^2 m_d \quad \text{(68)} \\
    c_d & = 2\sqrt{k_d m_d} \zeta_d \quad \text{(69)}
\end{align*}
Summary of Lecture 8

- **Symmetric Structures.**
  - Symmetry is required with respect to:
    - Geometry.
    - Boundary conditions.
    - Stiffness distribution.
    - Damping distribution.
    - Mass distribution.

  Computation efforts are reduced by approximately a factor $\frac{1}{4}$ for each symmetry plane. 3 Symmetry planes means a reduction factor of $\frac{1}{64}$. 
Vibrations due to Indirectly Acting Dynamic Loads

- Dynamic load $f(t)$ is not acting directly on the masses.
- Auxiliary degrees of freedom are introduced specifying the displacement of the load.

The resulting equations of motion is obtained on the standard form:

$$M\ddot{x} + C\dot{x} + Kx = Kd_0 f(t)$$

- $d_0$ : Vector of flexibility coefficients at the degrees of freedom due to a unit load at the indirectly acting load.